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ABSTRACT

This teaching experiment investigated variables which affect how well children learn from manipulative aids and how to use such aids in teaching mathematics. Five small groups from one second-grade classroom were stratified by achievement and taught using (1) counting sticks; (2) Dienes blocks; (3) abacus; (4) all three materials, or (5) counting sticks and unifix cubes, Dienes blocks and graph paper, and the abacus and colored chips. Two- and three-digit numeration and addition and subtraction with and without regrouping were presented in 22 lessons involving manipulative, picture, and symbolic phases. Lesson plans and reactions of pupils are included, in addition to data from evaluation interviews recorded on videotape and scored later. The abacus appeared to be less effective for teaching two-digit numeration than were blocks or sticks, but the three single embodiments appeared equally effective for teaching the other topics. Multiple embodiments were "superior" for developing two-digit numeration, but "not superior" for the other topics. Use of materials was more effective than non-use of materials (by a control group). Levels of behavior were listed with an interpretation of the findings. (MS)

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Teaching Experiment: The Effect of Manipulatives in Second Graders' Learning of Mathematics

Volume I

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FOREWORD

Ed Begle recently remarked that curricular efforts during the 1960's taught us a great deal about how to teach better mathematics, but very little about how to teach mathematics better. The mathematician will, quite likely, agree with both parts of this statement. The layman, the parent, and the elementary school teacher, however, question the thesis that the "new math" was really better than the "old math." At best, the fruits of the mathematics curriculum "revolution" were not sweet. Many judge them to be bitter.

While some viewed the curricular changes of the 1960's to be "revolutionary," others disagreed. Thomas C. O'Brien of Southern Illinois University at Edwardsville recently wrote, "We have not made any fundamental change in school mathematics."[1] He cites Allendoerfer who suggested that a curriculum which heeds the ways in which young children learn mathematics is needed. Such a curriculum would be based on the understanding of children's thinking and learning. It is one thing, however, to recognize that a conceptual model for mathematics curriculum is sound and necessary and to ask that the child's thinking and learning processes be heeded; it is quite another to translate these ideas into a curriculum which can be used effectively by the ordinary elementary school teacher working in the ordinary elementary school classroom.

Moreover, to propose that children's thinking processes should serve as a basis for curriculum development is to presuppose that curriculum makers agree on what these processes are. Such is not the case, but even if it were, curriculum makers do not agree on the implications which the understanding of these thinking processes would have for curriculum development.

In the real world of today's elementary school classroom, where not much hope for drastic changes for the better can be foreseen, it appears that in order to build a realistic, yet sound basis for the mathematics curriculum, children's mathematical thinking must be studied intensively in their usual school habitat. Given an opportunity to think freely, children clearly display certain patterns of thought as they deal with ordinary mathematical situations encountered daily in their classroom. A videotaped record of the outward manifestations of a child's thinking, uninfluenced by any teaching on the part of the interviewer, provides a rich source for conjectures as to what this thinking is, what mental structures the child has developed, and how the child uses these structures when dealing with the ordinary concepts of arithmetic. In addition, an intensive analysis of this videotape generates some conjectures as to the possible sources of what adults view as children's "misconceptions" and about how the school environment (the teacher and the materials) "fights" the child's natural thought processes.

The Project for the Mathematical Development of Children (PMDC)[2] set out

[1]"Why Teach Mathematics?" The Elementary School Journal 73 (Feb. 1973), 258-68.

[2]PMDC is supported by the National Science Foundation, Grant No. PES 74-18106-A03.

to create a more extensive and reliable basis on which to build mathematics curriculum. Accordingly, the emphasis in the first phase is to try to understand the children's intellectual pursuits, specifically their attempts to acquire some basic mathematical skills and concepts.

The PMDC, in its initial phase, works with children in grades 1 and 2. These grades seem to comprise the crucial years for the development of bases for the future learning of mathematics, since key mathematical concepts begin to form at these grade levels. The children's mathematical development is studied by means of:

1. One-to-one videotaped interviews subsequently analyzed by various individuals.

2. Teaching experiments in which specific variables are observed in a group teaching setting with five to fourteen children.

3. Intensive observations of children in their regular classroom setting.

4. Studies designed to investigate intensively the effect of a particular variable or medium on communicating mathematics to young children.

5. Formal testing, both group and one-to-one, designed to provide further insights into young children's mathematical knowledge.

The PMDC staff and the Advisory Board wish to report the Project's activities and findings to all who are interested in mathematical education. One means for accomplishing this is the PMDC publication program.

Many individuals contributed to the activities of PMDC. Its Advisory Board members are: Edward Begle, Edgar Edwards, Walter Dick, Renee Henry, John LeBlanc, Gerald Rising, Charles Smock, Stephen Willoughby and Lauren Woodby. The principal investigators are: Merlyn Behr, Tom Denmark, Stanley Erlwanger, Janice Flake, Larry Hatfield, William McKillip, Eugene D. Nichols, Leonard Pikaart, Leslie Steffe, and the Evaluator, Ray Carry. A special recognition for this publication is given to the PMDC Publications Committee, consisting of Merlyn Behr (Chairman), Thomas Cooney and Tom Denmark.

Eugene D. Nichols
Director of PMDC

PREFACE

The statement "Use manipulatives to teach mathematics" has been repeated so often and in so many different contexts that there is danger that the statement will begin to be accepted to the point where the use of manipulatives in teaching mathematics appears to be a panacea. There are, of course, no panaceas in the teaching of mathematics; yet, it is apparent that there are more questions that need to be answered about the why and how of using manipulatives to facilitate the learning of mathematics than most writers recognize.

While it may seem intuitively obvious that using manipulative aids to build a concrete conceptual base will in the final analysis help children to associate concepts and the symbolization of concepts in a meaningful way, it seems apparent that the "gap" between children's ability to perceive mathematics through manipulatives and their ability to associate symbolism with the concept is great. The question of how this gap is narrowed and finally closed is a problem that has had very little investigation.

When children first start school, they come with certain intuitive notions about mathematics. Whether or not these intuitive notions are tapped and built upon by instruction in our schools is an open question. This is true because we know very little about what these intuitive notions of mathematics are. Children have various finger manipulation strategies for doing addition, for example. Teaching practices frequently discourage children from using these strategies, and indeed, seldom are these strategies extended and a relationship between children's intuitive strategies and school strategies developed.

This publication is intended to share with the reader the information obtained from a teaching experiment which dealt with the question, "What are some important variables which affect how well children learn from manipulative aids?" Information about the relative effectiveness of Dienes blocks, counting sticks, and an abacus for the teaching of place value concepts to second grade children was investigated. Also of interest in the experiment was the question of whether systematic use of all three of the manipulatives would prove to be more effective for learning these concepts than just one manipulative.

The approach used in this study was that of a teaching experiment. The concept of a teaching experiment employed was that of gaining practical and anecdotal data in a teaching-learning situation. The teaching involved a teacher working with a small group of children, rather than a normal-sized class. Because of the small groups, statistics presented in this study's statistical data must be interpreted with caution. Of more interest are the questions and hypotheses which are suggested by the investigation.

The research reported in Volumes I and II was an attempt to gain some insights about the question of what are significant variables related to how to use manipulatives in teaching mathematics to children. The reader

will soon observe that the research method employed was different from that of traditional education research. Very small groups of children were involved in what was considered a teaching experiment. This concept of a teaching experiment represents a first (or second) step in a research effort. The research reported herein is not hypothesis testing of the familiar research tradition. Instead, it is hypothesis generating--more of an attempt is made at clarifying problems for further investigation than at answering pre-stated questions.

The reader of the volumes will find data presented in various forms; a great deal more "raw data" is presented than is ordinarily done in a research report. This raw data is presented in such diverse forms as raw scores on tests, both written and clinical interview tests, summaries of child responses extracted from daily logs kept by teachers of the small groups, and, finally, a number of case studies of children involved in the experiment.

Volume I contains sections which describe the rationale and conduct of the experiment in detail. In addition, Volume I includes information about the results of the investigation. Volume II consists entirely of the fifteen case studies conducted by the group teachers. As background for Volume II, the reader should refer to Chapters I and III of Volume I.

Many individuals contributed to the conceptualization of this study. The contribution made by the PMDC Advisory Board, Staff, and Evaluator, who reacted to the initial proposal is gratefully acknowledged. Special thanks are due Cynthia Clarke, Patricia Campbell, Stewart Wood, Judy Voran, and Ella Barco, who served as group teachers in the teaching experiment, and Max Gerling, who supervised the videotaping of lessons and interviews. Thanks are also due the project administrative assistant, Janelle Hardy, publications editor, Maria Pitner, and typists, Mary Harrington, Julie Rhodes, and Joe Schmerler.

MERLYN BEHR
Principal Investigator

Professor Behr is on leave of absence from Northern Illinois University.

I. INTRODUCTION

RATIONALE FOR THE STUDY

Z.P. Dienes suggests that abstraction of a mathematical concept occurs when a child has observed or discovered the commonality among several embodiments (manipulative aids) of the concept. This suggests that more than one embodiment may be necessary for children to acquire abstract mathematical concepts. Another idea related to Dienes' theory is that embodiments employed for learning a concept should vary according to mathematical properties inherent in the embodiments. For example, the three embodiments--counting sticks, Dienes blocks, and an abacus--vary in that place value numeration of whole numbers is embodied by counting sticks based on the numerosity of the sticks, by Dienes blocks based on the comparative lengths of the blocks, and by the abacus by an assigned value according to color or position.

A number of writers and researchers have raised questions about the appropriate use of embodiments in teaching and learning which are investigated through experimentation. Many such questions can be put in the context of Dienes' theory. For example, how many embodiments should be used? Which embodiment(s) is (are) most effective for a given concept and why? Is the order in which embodiments are introduced significant? Does the complexity of the skill or task to be learned interact with the number or type of embodiment(s)? Does the number or type of embodiment(s) interact with characteristics of children?

PURPOSE OF THE STUDY

The purpose of this clinical teaching experiment was to gather and record largely qualitative anecdotal evidence to aid in hypothesis formation with regard to:

1. The differential effect of three different uni-embodiment learning environments on second grade children's skill (ability to demonstrate or compute) and understanding (ability to explain processes) of 2- and 3-digit numeration.

2. The differential effect on second grade children's skill and understanding of 2- and 3-digit numeration of uni- vs. multi-embodiment environments.

3. The differential effect of three different uni-embodiment environments on second grade children's learning (ability to compute) and understanding (ability to explain processes) of 2-digit addition and subtraction.

4. The differential effect on second grade children's learning (ability to compute) and understanding (ability to explain processes) of 2-digit addition and subtraction of a uni- vs. multi-embodiment environment.

5. The differential effect of uni- vs. multi-embodiment environments on second grade children's ability to transfer their knowledge of 2-digit addition to 3-digit addition.

6. The differential effect of uni- vs. multi-embodiment environments on second grade children tested on the impulsive-reflexive measure (Kagan, 1965), (i.e., investigate whether an ATI exists between cognitive style and method of instruction).

7. The differential effect of children's learning and understanding of 2-digit addition when required to transfer their manipulative skill on one embodiment to another within three different levels of the mathematical variability principle.

8. The differential effect of three different uni-embodiment learning environments on second graders' ability to transfer skill and understanding of 2-digit addition and subtraction algorithms with regrouping to 3-digit addition and subtraction with regrouping.

9. The differential effect of uni-embodiment and multi-embodiment learning environments on second graders' ability to transfer skill and understanding of 2-digit addition and subtraction algorithms with regrouping to 3-digit addition and subtraction with regrouping.

II. RELATED RESEARCH[1]

The related research in the area of the use of manipulative aids in the teaching and learning of mathematics falls into two broad categories. One category of the research on manipulatives is topic or content oriented. The main thrust of these investigations has been to determine the role of manipulatives or the relative effectiveness of a manipulative or non-manipulative approach on the learning of a specific grade level.

In a study with three- and four-year-olds, Williams (1969) found that environmentally deprived children were able to learn basic concepts in linear measurement using a manipulative learning aid that he designed. The aid enabled the child to see in concrete terms the size and space relations used in linear measurement concepts of one-half, one-fourth, and one-eighth.

Pan balances and mathematical balances were used to instruct first grade pupils in the basic addition facts in a study by Dashiell and Yawkey (1974). Comparing these two manipulatives, they found the mathematical balance to yield significantly better results on a standardized test on addition facts.

Steffe and Johnson (1970) investigated the problem-solving abilities of first graders. They gave a 48-item test and compared performances of children who were allowed free use of manipulatives in solving the problems with children who had no manipulatives available. They found that the group with manipulative aids performed significantly better than the deprived group on all but one of the eight problem types in all but one of the four ability groups.

[1]A study of related research appears in Gerling, Max, and Stewart Wood Literature review: Research on the use of manipulatives in mathematics learning. PMDC Technical Report No. 13. Tallahassee, Florida: Florida State University, 1977.

In another study with first graders, Prindeville (1971) gave 24 supplementary lessons on place value, order of numbers to 100, and two-place addition and subtraction. The two classes using manipulative materials and language training significantly outperformed the control group in the posttest and retention test.

DeFlandre (1974) field tested a unit on place value numeration with second, third, and fourth grade classes. He concluded that this unit, using manipulative aids and multiple embodiments, was an effective way to teach place value numeration systems and express these abstractions in symbolic form. He also concluded that children who studied this unit could not transfer the concept of place value numeration systems to addition and subtraction at a symbolic level.

In a study involving the recall of basic multiplication facts, Babb (1975) compared three treatments given to second graders: a textbook approach, a manipulative material approach, and an imagery-mnemonic approach. He found the adjusted mean recall score for the manipulative group to be significantly higher than that for the imagery group, but no significant differences in final recall and comprehension were found between the manipulative and textbook approaches, although the manipulative approach yielded a significantly higher attitude score.

In another study of instructing third graders in multiplication using four different treatments, Moody, Abell, and Bausell (1971) found no significant differences in any of the four treatments: activity-oriented, rote, rote-word problem, and control. The validity of this study was strongly questioned by Holz (1972) in his critique of the article reporting their study.

Nichols (1971) compared two methods of instruction in multiplication and division with third graders. She found significant differences favoring a manipulative approach with pupil discovery over a semi-concrete, abstract approach in all the 16 hypotheses of the study.

Several studies involved various approaches to the teaching of fractions. In a study by Brown (1972), four approaches to teaching equivalent fractions to fourth grade pupils yielded the results that a textbook approach was inferior to three other approaches: the textbook with film, textbook with manipulatives, and textbooks with film and manipulatives. He also found the textbook-film-manipulative approach to yield significantly higher mean scores than the other three groups.

Bisio (1970) investigated instruction in addition and subtraction of like fractions in grade five using three treatments: no manipulatives, teacher-demonstrated manipulatives, and teacher and student use of manipulatives. He concluded that the demonstration use of manipulative materials appeared as effective as use by the student and was better than non-use of the manipulatives.

Green (1969) compared the effects of two instructional approaches and two instructional materials on the teaching of multiplication of fractions with fifth graders. She concluded that diagrams and manipulative aids were equally effective in the learning of multiplication of fractions, and an attitude test showed that pupils liked the diagram approach better than the manipulative approach.

Purser (1973) investigated the relation of manipulative activities, achievement, and retention in teaching fractions and decimals to seventh grade pupils. He found the group using manipulative activities had significantly higher scores on posttests and retention tests than the group using only paper-pencil type activities. Bledsoe, Purser, and Frantz (1974) reported similar results in the Journal for Research in Mathematics Education the following year.

Coltharp (1968) compared the effectiveness of a concrete and an abstract approach in teaching integer arithmetic to sixth graders. He found no significant differences and concluded that pupils taught addition and subtraction of integers by an abstract, algebraic approach achieved as well as those taught by a concrete, visual approach.

Bring (1971) investigated the effects of varying concrete activities on the achievement of fifth and sixth graders in learning topics of geometry. He found that instruction using concrete activities produced higher achievement, higher interest, and lower anxiety than instruction without concrete activities.

III. DESIGN AND CONDUCT OF THE EXPERIMENT

SCHOOL

The experiment was conducted in a public elementary school in a city of 35,000 in the Southeast. The area served by the school includes many families of a low socioeconomic status. About two-thirds of the student body were from these families. Also included in the area served by this school is the married student housing complex for a large university. The school has a rather bimodal distribution of children according to ability.

TEACHING GROUPS

The treatment (teaching) groups for the study were formed by a rank-order random selection procedure. This was accomplished by ranking the 30 children within an intact second grade class according to their total score on the KeyMath Diagnostic Arithmetic Test. Six strata of 5 children were formed which also indicates 6 levels of ability as measured by KeyMath total scores. Each of the children within each of the six strata was randomly selected for one of the five experimental teaching groups. This resulted in 5 teaching groups which were comparable in ability as indicated by KeyMath total scores. Each of the 5 teaching groups ranged in ability from children with very low KeyMath scores to children whose scores were quite high. Unfortunately, a number of children in some groups moved from the school during the year; this reduced the number of children in some groups and also destroyed the equivalence of the groups according to ability. A control group was selected within the same school from among second graders who were assigned to work within the 3rd grade class. The means, standard deviations, and ranges of the KeyMath scores and Otis-Lennon Mental Ability (IQ) Test scores are given in Table 1.

For purposes of analysis only the data from children who remained in the teaching groups for the entire year were used.

Table 1

KeyMath and Otis-Lennon IQ mean score, standard deviations, and range by groups.

		KeyMath			Otis-Lennon		
Group	N	Mean	Std. Dev.	Range	Mean	Std. Dev.	Range
U1	5	49.40	14.18	28 - 67 (0.7 - 2.2)*	83.00	10.84	69 - 109
U2	5	50.60	20.18	25 - 79 (0.5 - 2.5)	93.40	9.21	88 - 94
U3	6	51.00	14.53	37 - 74 (1.1 - 2.4)	94.50	10.27	73 - 131
M	4	67.25	18.76	51 - 93 (1.7 - 3.0)	99.25	21.16	75 - 124
2M	4	48.00	14.54	35 - 66 (1.0 - 2.2)	85.25	8.92	75 - 96
C	5	78.00	13.30	67 - 100 (2.2 - 3.2)	+	+	+

*Grade equivalents of the total raw scores.

+Otis-Lennon IQ scores were not available for the control group.

It is evident from Table 1 that the groups were not equivalent in ability as indicated by KeyMath and Otis-Lennon IQ scores after some children had moved.

TEACHING MATERIALS

Detailed teaching materials were written by the investigator. These materials are described in a later section of this report. The essential difference among the teaching materials for the groups U1, U2, U3, M, and 2M was the embodiment(s) [manipulative aid(s)] used for teaching and learning in that group. These were as follows:

U1--counting sticks

U2--Dienes blocks

U3--abacus

M--counting sticks, Dienes blocks, and abacus

2M--counting sticks and unifix cubes, Dienes blocks and graph paper, and the abacus and colored chips

Appropriate manipulatives were purchased so that each child and teacher had materials for his/her own use.

PURPOSE AND PHILOSOPHY OF THE TEACHING GROUPS

The embodiments (manipulative aids) used in this teaching experiment can be considered to represent three categories of manipulatives. These categories can be defined according to the way in which the embodiment represents the grouping of objects so that base ten numeration is reflected. Counting sticks represent a category of embodiments which can be characterized in that a group of ten (1 ten) is divisible into ten sticks (10 ones). This distinguishes counting sticks from Dienes blocks. In the Dienes blocks embodiment the long (1 ten) is exactly equal to ten units placed end-to-end. However, in this system the long (1 ten) is fixed and cannot be divided or separated into ten units (10 ones). Another category of embodiments for base ten numeration is characterized by an abacus. The idea of 1 ten being equal to 10 ones is exemplified in this embodiment only by a value assignment to the successive rods on the abacus. Thus the abacus is quite different in this respect from either the counting sticks or Dienes blocks.

Teaching groups U1, U2, and U3 were formed in order to gain information about whether second graders learn mathematical concepts more readily by using any one of these three embodiments over another one. Teaching group M was formed in order to gain information about whether or not second graders learn mathematical concepts more readily by using one embodiment from each of the three categories rather than one from any of the three.

The teaching philosophy in groups U1, U2, U3, and M was very similar. It was a demonstration-discussion-practice mode. The group teacher would demonstrate how the embodiment was to be used to represent a concept or perform an operation, after which the children practiced this representation with guidance from the teacher. Thus the children's learning could be characterized as having been acquired through imitative behavior. That is not to say that children were not encouraged to be creative with the manipulatives, but the general mode of instruction was basically demonstration and discussion followed by practice. The original purpose of the 2M group was related to a learning theory espoused by Merrill Wittrock (1974). According to this theory our second graders' learning of mathematical concepts would have been facilitated by requiring the children to "generate new knowledge"--that is, by applying what Wittrock refers to as generative processing. It was intended that the teaching philosophy for the 2M group would incorporate this notion of generative processing. This was to have been done by having the children extend their imitative-behavior learning about the representation by one manipulative in a category to another manipulative in the same category. This was required for a pair of manipulatives in each of the three categories as follows: From counting sticks to unifix cubes, from Dienes blocks to graph paper, and from the abacus to counting sticks.

In retrospect it appears that no real generative processing took place, at least not beyond the first lesson. It appears that once children made the extension from one manipulative in a category for one concept, thereafter the manipulatives within a category became interchangeable. In this sense the teaching group really became a group using six embodiments, two embodiments in each of three categories.

IV. THE TEACHING MATERIALS AND STUDENT REACTIONS

The instructional materials for each of the groups were developed to teach the standard number related concepts -- place value, addition, and subtraction -- in grade 2. The lessons were developed in groups according to the following topics

1. 2-digit numeration - 7 lessons
2. Addition without regrouping - 4 lessons
3. Subtraction without regrouping - 4 lessons
4. Addition with regrouping - 3 lessons
5. Subtraction without regrouping - 2 lessons
6. 3-digit numeration - 2 lessons

No attempt was made to have a lesson correspond to one day of instruction, rather each lesson was designed to reflect a small unit of instruction. Each lesson in the 22 lesson sequence was written to begin instruction at the enactive (manipulative) phase and continue it through the iconic (picture) and symbolic phases.

During the enactive phase of instruction children were taught how to perform certain tasks using the manipulative(s) of the group. All question-answer interaction between the teacher and children was either in the oral-manipulative order or the manipulative-oral order. No symbolism or pictures were used in this phase.

During the iconic phase children were to use manipulatives to suggest a static situation or sequence depicted by a picture (or pictures) of the corresponding manipulative, and conversely to choose and sequence picture(s) to reflect a manipulative display. The characterizing feature of the iconic phase was that the question-answer interaction between teacher and children was oral-picture, manipulative-picture, picture-manipulative, or picture-oral. No symbolism was employed during the iconic phase.

During the symbolic phase children used manipulative aids to solve problems presented symbolically and orally, or gave symbolic and oral answers or explanations to situations presented orally, with manipulatives or with pictures. An attempt was made to involve the teacher and children in each of the following question-answer orders: symbolic-manipulative, symbolic-picture, symbolic-orally, manipulative-oral, manipulative-symbolic, picture-symbolic, and oral-symbolic.

The lesson materials were organized into units which included very

detailed guidance to the group teacher for demonstrations and questions to be presented to the children. The lesson materials for class instruction were written in a two-column format. One column was headed "DO" and the other "By using questions, discussion, motions, direct commands, or whatever is comfortable to you, DIRECT THE CHILDREN TO." In the "DO" column were listed in detail situations the teacher would present to the children. In the other column, in detail, were given activities which the teacher was to direct the children to do. It was felt that this format provided maximum guidance to assure uniformity of instruction among the instructional groups, yet allowed the group teacher to adapt the instruction to his/her personality. The instructional materials for each treatment group were parallel in terms of the kind and order of the activities used.

Samples of the lesson materials are presented in Appendix A.

2-DIGIT NUMERATION

The broad objective of this sequence of lessons, lessons 1-7, was to extend the children's number concept to numbers in the range 10-99, numbers named by 2-digit numerals. In the lesson sequence, special emphasis was given to providing activities which would develop the following concepts associated with 2-digit numeration.

1. That 10 unit objects represent the same number as one 10-object. That is, that 10 ones is equal to 1 ten.
2. That numbers in this range (10-99) can be thought of as tens and ones. For example, that 24 is 2 tens and 4 ones.
3. That numbers in this range can be thought of as a multiple of ten and ones. For example, that 24 is twenty and 4 more.
4. That numbers in this range can be thought of as ones. For example, 24 can be thought of as 24 ones.

Lesson 1 introduced the manipulatives to the children. The lesson began with a period of free play with the manipulatives followed by structured activities leading to the notion that these objects could be used to represent numbers.

The broad objective of lesson 2 was to associate 2-digit numerals--oral and written--with manipulative and picture-of-manipulative displays for numbers and vice-versa. Activities were included which related to the four concepts listed in the previous section. For example, in one activity the teacher would display a set of unit objects, then arrange them in sets of ten and one set of fewer than ten and then replace the 10-sets by a representative of 1 ten. Interaction between the teacher and children dealt with questions about whether the number changed; that 12, for example, is 12 ones, 10 and 2 more, 1 ten and 2 ones. Children were also given the opportunity to manipulate the objects to reflect these different views of the same number. The numbers used in this lesson ranged from 10-32, with

most activities concentrated on numbers less than 30. The time taken for lesson 2 was about two days for the single-manipulative groups and six days for the multi-manipulative groups.

Notes from the logs kept by group instructors suggest that some children had difficulty counting a display with a representative of ten and some ones To point at the ten and count on--10, 11, 12 -proved to be difficult for some children; they needed to count the representative of ten and say, "1, 2, 3, ..., 10, 11, 12." It appeared that no children had difficulty trading one representative of 10 for 10 representatives of one. In all of the groups children had difficulty completing a statement like ___ tens and ___ ones to correspond to a manipulative or picture display; however, the children were able to tell how many tens and ones when this question was presented orally.

The broad objective of lesson 3 was to work on the number concept for numbers 10-20. Activities related mostly to counting activities and recording the numbers in order. Numbers were identified by the children orally in response to the teacher's manipulative displays. The counting sequence was shown in three ways: with manipulatives, by ordering pictures, and symbolically. Special emphasis was given to observation of the pattern that 10 is 1 ten and 0 more, 11 is 1 ten and 1 more, etc. As the single-manipulative groups neared completion of this lesson, it became apparent that the difficulty which children were having with exercises like the completion of

___ longs and ___ units

___ tens and ___ ones

to correspond with a manipulative display or picture was due, in part, to their inability to read the words longs, units, tens, and ones. Therefore a supplement to lesson 3 was written which gave practice on the reading of vocabulary appropriate for each group. The reading and vocabulary activities were conducted within the context of number activity. The length of time required for the lesson 3 supplement varied considerably among the groups. The vocabulary for the U3 group involved only the words tens and ones; whereas the 2M group was concerned with the words bundles, sticks, longs, units, rods, cubes, strips, squares, as well as tens and ones.

The activities of lesson 4 were related to the objective of developing rational counting by tens from 10 through 90. Children were given activities which required them to give orally and symbolically the numerals as the teacher displayed successively 1, 2, 3, ..., 9 representatives of 10, to give successive representative as the teacher gave the number names 10, 20, ..., 90 orally and symbolically, and to order pictures of manipulative display having 1, 2, ..., 9 representatives of ten.

In lesson 5 the actual consideration of place value and the value of digits in a 2-digit numeral was begun. The stated objectives for the symbolic phase of this lesson were:

1. Given a manipulative of a picture-of-manipulative display of a number like 64 to:
   a. Write the number 64,
   b. Ring the digit which tells there are 6 tens,
   c. Ring the digit which tells there are 4 ones.

2. Given a numeral such as 64 to:
   a. Give a manipulative or choose a picture-of-manipulative display for the number,
   b. Point to the objects or ring the part of the picture to show that the 6 in 64 means 60,
   c. Point to the objects or ring the part of the picture to show that the 4 in 64 means 4 ones.

Appropriate modifications of these statements to reflect only non-symbolic activity define the objectives for the enactive and iconic instructional phases. Activities were included for numbers of magnitude such as 92, 77, 38, 83, 10. During this lesson it was observed that children had considerable difficulty dealing with numbers of this magnitude at this time. Therefore it was decided to write a supplemental lesson--lesson 5S--to give children more experience with numbers by decades. Thus lesson 5S was written to essentially duplicate the activities of lesson 4 for numbers 20-30, 40-50, and 80-90. During the symbolic phase of lesson 5S the place value chart was introduced as an aid for writing 2-digit numerals.

The title for lesson 6 was "Place value, expanded notation, and 2-digit numerals 10-90." The stated objectives for the iconic instruction phase were as follows:

1. Given a pictured manipulative display for a number like 64 to orally express this number as 6 tens plus 4 ones, sixty plus four, and sixty-four.

2. Given any of three oral expressions--6 tens plus 4 ones, sixty plus four, or sixty-four--for a number like 64 to:
   a. Choose the correct picture-of-manipulative display,
   b. Give the other two oral expressions.

Appropriate modifications define the objectives for the enactive and symbolic instructional phases. Included in the worksheets for the symbolic phase of lesson 6 were exercises which required the children to compute statements:

___ tens + ___ ones

___ + ___

___

for picture-of-manipulative displays for numbers. Some children, particularly those of lower ability, would write 40 tens + 3 ones and some would write 403 for the number forty-three. At this point in the instruction

such written exercises as this continued to cause some difficulty for some children, probably due to a reading problem.

Lesson 7, the last lesson of this first lesson sequence, dealt with ordering numbers named by 2-digit numerals. The stated objectives for the three phases of instruction were as follows:

Enactive phase

1. If shown a manipulative display for numbers like 12 and 19, to
   a. Tell which display has more (fewer) objects,
   b. Tell which display represents the greater (lesser) number.

2. If read a statement like nineteen is less than twelve, to show that this is true or false using manipulatives and a 1-1 correspondence idea.

Iconic phase

1. If given a picture-of-manipulative display for two 2-digit numbers, to
   a. Say the numbers shown,
   b. Tell which display has more objects,
   c. Tell which display has less objects,
   d. Tell which display shows the greater number,
   e. Tell which display shows the lesser number,
   f. Say the appropriate "is less than" sentence,
   g. Say the appropriate "is greater than" sentence.

2. Given a set of pictures-of-manipulative displays, to sequence them in increasing (and decreasing) order.

Symbolic phase

1. Given a picture-of-manipulative display for two 2-digit numbers x and y ($x < y$) to
   a. Write the numerals for the numbers shown,
   b. Order the pictures left to right in less to greater order,
   c. Say the sentence x is less than y,
   d. Construct with word and symbol cards the sentence x is less than y,
   e. Construct with symbol cards the sentence $x < y$,
   f. Construct with word and symbol cards the sentence y is greater than x,
   g. Construct with symbol cards the sentence $y > x$,
   h. Write $x < y$,
   i. Write $y > x$.

2. Given two 2-digit numerals, to
   a. Ring the numeral which "is less,"
   b. Box the numeral which "is greater,"
   c. Complete the appropriate $<$ sentence,
   d. Complete the appropriate $>$ sentence.

LOG EXCERPTS

The following excerpts taken from summaries of the group instructors' logs are given in an attempt to present children's reactions to some of the materials on 2-digit numeration.

Lesson 2

Joe Berny, Cubby, Brett, and Phil were able to count various stick displays and quickly learned that a bundle consisted of ten sticks.

Carrie had difficulty counting stick displays because she was unable to do rational counting.

Cubby had difficulty ordering pictures (to correspond to a demonstrated sequence of manipulations of the sticks). Carrie had difficulty ordering pictures because she is unable to do rational counting.

Carrie . . . is beginning to learn rational counting and knows that there are ten sticks in a bundle.

Joe Benny . . . called [illegible] ten, 10 tens.

All students had difficulty going from the step ___ tens and ___ ones to ___ and ___ on answer sheet.

Brett showed 39 by placing 4 bundles and removing 1 stick from one of the bundles.

Carrie is beginning to employ rational counting and has learned to separate sticks into tens and ones.

Celia, Washington, and Duke prefer to show numbers like 14 with long and units rather than all units.

Duke has to count the units in 2 longs in order to answer how many; everyone else knows "20."

In counting pictures showing a long and some units, all children except Duke count on from 10.

Children cannot observe the entire trading sequence and then put pictures in order; they <u>can</u> find picture for each step of the process as it is being done.

Duke and Celia have trouble with the lines "___tens and ___ones"; they usually fill in <u>10</u> tens and <u>4</u> ones (for a display for 14)

For Duke and Celia, the printed word "ten" doesn't seem to mean a thing.

Trading (ten beads on the) ones for a (bead on the) ten was easy for everyone.

Ordering pictures (of the abacus to show a trading sequence) was easy; kids ordered them before I could go through the sequence.

They (two of the children) wrote 12 as 10 tens and 12 ones.

For "___ and ___ ones" Annie put, for 14, 4 and 10 ones; didn't see the pattern.

Paul, Kate, and Annie frequently wrote 10 tens instead of 1 ten.

Annie cannot read (give an oral response for) any 2-digit numerals, but she can show it on her abacus if she sees it written.

Alex and Karl have trouble ordering pictures to show manipulations involved in bundling and trading.

Children have no trouble imitating stick (block) pictures with block (stick) manipulations.

On examples where the number is greater than 15, all children have trouble locating sufficient sticks or units without losing their "place" in the counting sequence.

Calvin and April combine use of sticks and blocks to display amounts.

Lessons 3 and 3S

When asked to show 20, Brett counted out 20 sticks and then made 2 bundles of 10.

When asked to count from 10 to 20, Joe Benny started at 1.

Phil was only one who knew what to do to order pictures from 10-20; he found 10, searched for 11, 12, etc. Other children put out all pictures and started shifting them back and forth with no noticeable plan.

Opie had difficulty reading words: bundle(s), stick(s), ten(s), and one(s); by the end of the period, however, all but Carrie had learned words and could display correct stick display in response to written stimuli.

In ordering pictures, Celia and Washington counted the units (ignoring the longs in each picture) while Duke counted the total number in each picture by 1's.

Duke now counting by tens and ones correctly.

When asked how much 2 longs and 3 units is without a block display, no one could answer.

In writing numerals from 11 to 20, Celia focused on units again, anticipated by displays, wrote 17, 18, 19, 10.

Girls preferred showing teen numbers on ones rod only, no tens.

Kate and Paul caught on to "short-cut"--just add another one to show next larger number instead of clearing the abacus.

On worksheet with form. ___ and ___ ones

___ tens and ___ ones

Claire and Kate wrote 3 digits on botton line, 203 for 23, etc.

Alex and Richard can fill in ___ bundles and ___ sticks but need verbal cues to fill in ___ tens and ___ ones.

Alex had trouble ordering pictures and correcting from 12 to 15. Task challenging for all.

They had difficulty going from 2 longs + 3 units to 20 + 3. This is partly due to a reading problem. They seem to know that 2 longs is 20, but just don't know where to write it.

Still problems with ___ tens and ___ ones. It is obvious that they "know" more than they are able to symbolize.

Tommy counted a bundle and 9 sticks by mistake when trying to show 17. Instead of recounting he took 2 away.

Lesson 4

All children were able to tell how many when 1-9 bundles were displayed, and all could display correct number of bundles for numbers 10-20 when given orally in sequence.

Phil, Opie, and Joe Benny ordered pictures 10-90 correctly and knew what to do immediately.

All students can count longs by tens to 90 and answer questions like "How many tens in 90?" with their block displays in front of them.

All children can find picture (of 7 longs) that shows 70 and answer the questions "How many tens in 70?" and "How much is 7 tens?"

Worksheet with "9 tens = ___ and ___ tens = 30" caused trouble.

When asked to tell how many in a display (e.g., 40) Karl and Alex first count how many tens (1, 2, 3, 4 bundles or longs) then translate it (40).

Reviewed reading words [long(s), (bundle(s), unit(s), stick(s), ten(s), one(s)]. Alex literally wilted--slid under the table. But he was fine, interested, working on Lesson 4 before the word cards.

When I read "thirty," Mary wrote 30. Later when I wrote 30 and asked her to display she asked "What's that number?"

Lessons 5 and 5S

All children started the ordering of pictures from 20-30 in the following order: 21, 22, ..., 30 and became puzzled as to where to put the 20.

Carrie and Cubby still had trouble ordering 20-30 sequence.

All were able to order 40-50 sequence. However, Phil, Cubby, and Carrie had trouble ordering 80-90 sequence.

All seemed to be able to tell how many tens and ones were in a 2-digit number but all had difficulty determining how many ones in a particular "2-digit number."

..., they had difficulty determining the value of a ringed digit (tens or ones).

First experience with numbers greater than 50.

When counting a block display I have made, Duke and Washington tend to count both longs and units by 10's; Washington catches himself and corrects, but Duke can't. Others do OK.

Duke and Celia continue to have trouble counting blocks in my display --they get going by 10's (by rote) and don't shift to ones appropriately.

In ordering pictures in a decade they beginning and end (e.g., 40 and 50) were troublesome to all children.

Kids wanted to say, to answer how many tens and how many ones, "20 and 3 ones," etc. They couldn't say "2 tens," but would count and say "20."

Everyone did well using (place value chart) and abacus picture, even Annie, though she couldn't read what she wrote.

Everyone could find picture for a numeral. Everyone had trouble converting 3 tens to 30, usually wrote only first digit, without zero.

Hard to break rote habit of 10, 20, 30, 40, ... rather than counting 10, 20, 21, 22, ...

Children had shown 85 with sticks; then I asked them to show 81 with sticks. Karl and Alex put the 85 sticks back in their boxes and recounted 81. Karl hesitates and counts over and over again when "going" from tens to ones; e.g., (for 45) "10, 20, 30, 40, ... 10, 20, 30, 40" over and over again until he finally says, "10, 20, 30, 40, 41, 42, 43, 44, 45."

Alex has extreme difficulty "going from tens to ones" when counting.

Alex seems to fill in tens | ones readily enough but has trouble writing the numerals if this framework is not provided.

Abel could not count 44 shown on Tommy's abacus. He counted 10, 20, 30, 40, 21, 22, 23, 24. She put two red chips and 10 white chips to show 30. When asked she said there are 2 tens in 30. After trading 10 white chips for one red, she said there are 3 tens in 30--seeing no contradiction.

Lesson 6

All children seem to have understood how to tell orally from a stick display the following:

____ tens + ____ ones

____ + ____

____

Carrie referred to 3 bundles and 2 ones as 10, 20, 30, 40, 50.

For a display of 4 bundles and 4 sticks Brett wrote 4 + 4 instead of 40 + 4.

For a display of 6 longs and 7 units Celia and Duke are more able to say "60 + 7" than to say "6 tens + 7"; Washington says "6 tens + 7 ones" easier; Susan and David can handle both in describing blocks.

Had difficulty eliciting response of 40 plus 3 for 43; 4 tens plus 3 came easier.

ADDITION FOR NUMBERS NAMED BY 2-DIGIT NUMERALS--NO REGROUPING

The broad objective of this sequence of lessons--lessons 8-11--was to develop the addition algorithm for addition of numbers named by 2-digit numerals. This sequence of lessons was limited to addition without regrouping. At the enactive level special emphasis was given to review and extend the concept of addition as the union of sets. The sequence of lessons were designed to present the development of the concepts associated with the addition algorithm in the following order.

1. Review basic facts with sums less than 10.

2. Extend this skill to compute basic facts to compute sums of two numbers both of which are multiples of ten.

3. Compute sums of numbers which are multiples of ten and numbers less than 10.

4. Computation of the sum of two numbers both named by 2-digit numerals without regrouping.

5. Computation of sum of two numbers, one named by a 2-digit numeral and another named by a 1-digit numeral, without regrouping.

The "expanded" addition algorithm which was taught is illustrated by the following problem.

$$\begin{array}{r} 24 \\ +31 \\ \hline 5 \\ 50 \\ \hline 55 \end{array}$$

Manipulative support was developed for this algorithm at the enactive level by having the children join ones and tens but keeping them separated and

then having them perform a second joining to determine "how many altogether." Similarly at the symbolic level, children were required to write the algorithm in this form as well as to show each step with manipulatives.

Lesson 8 was titled "Addition Review facts, tens and tens, tens and ones." The stated objectives for this unit were:

1. To give oral addition sentences which correspond to object manipulations which suggest addition for:
   a. basic facts, sums to nine,
   b. tens and tens, sums to 90,
   c. tens and ones, sums to 99.

2. To give object manipulations for orally presented addition sentences for:
   a. basic facts, sums to nine,
   b. tens and tens, sums to 90,
   c. tens and ones, sums to 99.

The objectives for the iconic and symbolic phases were the same except the stimulus or the expected response modes were pictural or symbolic.

Lesson 9 was titled "Place value review" and was presented at the symbolic level only. The stated objectives for this lesson were:

1. When presented an object or picture-of-object display for a number 10-99 to write the number of tens and ones and write the 2-digit numeral.

2. When shown a 2-digit numeral to show the appropriate object display or choose an appropriate picture.

Lesson 10 was titled "Addition: 2-digit and 2-digit without regrouping." In this lesson an "Addition-Form-Board" was introduced. The purpose of the form board was to provide a structure on which the children would display objects to represent each addend and also to union of the sets of objects to represent the sum. The Form Board was constructed from light cardboard. During the enactive phase of the lesson teacher demonstrations showed the children how to display addends on the Form-Board, how to move the objects together and place them into the space suggested for the "sum-set." Children were allowed to be flexible about first joining tens or ones. However, if children were observed to rigidly first join tens or ones they were encouraged to rigidly join the ones first. The stated objectives for the three phases of instruction for this lesson were as follows.

Enactive phase

To use objects to find the sums of two numbers named by 2-digit numerals.

Iconic phase

1. Given a pictured object display of two 2-digit addends to show the

same with the objects and use the objects to complete the addition.

2. When given an addition problem orally to choose and sequence pictures to show the addition.

3. Given an object display of two 2-digit addends to find the picture-of-objects which shows the sum.

4. Given a picture-of-objects display of two 2-digit addends to —
   a. tell (orally) what addition problem is shown,
   b. display the answer with objects.

Symbolic phase

1. When shown a 2-digit 2-addend addition problem in symbolic form to
   a. read the problem,
   b. display the addends on the Form-Board,
   c. perform the addition with objects,
   d. give the answer orally,
   e. write the answer.

2. When shown a 2-digit 2-addend addition problem in iconic or enactive form to
   a. symbolize the addends,
   b. solve the problem with the objects,
   c. write the answer to the problem.

During the first part of the symbolic phase of lesson 10 use of a place-value-chart w: emphasized; later in this lesson, illustrations and worksheets were presented without the place-value-chart.

The last lesson of the sequence--lesson 11--was titled "Addition: 2-digit and 1-digit without regrouping." The stated objectives of this lesson were the same as for lesson 10 with appropriate modification for the different content.

The order lesson 10 followed by lesson 11--2-digit and 2-digit addition, the 2-digit and 1-digit--is not the usual order followed in most textbook series. It was conjectured that if children learned the algorithm for two 2-digit addends that the procedure would transfer to the 2-digit and 1-digit type problems. This appeared to be true. Few children had difficulty learning the algorithm for 2-digit and 1-digit addends after having learned the more general algorithm for two 2-digit addends.

LOG EXCERPTS

The following excerpts taken from summaries of the group instructor's logs are given in an attempt to present children's reactions to the materials.

Lesson 8

For a display of 2 sticks and 7 sticks (to suggest two addends), Carrie

said the answer was 27. All others were able to show addition of number pairs by displaying appropriate sticks and verbalizing what they had done.

Cubby chose pictures for 3 bundles and 2 sticks for 3 plus 2 equals 5.

Carrie wrote 70 + 10 = 80 (for picture-of-manipulative display) for 70 + 1 = 71.

There was considerable longs/units confusion on this first day. Example: 5 + 4 = 90 (David with longs--he corrected himself), 3 longs and 5 longs = 8 (Duke--he says nothing wrong with this). Celia and Duke used longs to illustrate 6 + 2 = 8. Celia used units to show 2 + 7 and said the total was 27. In general children avoided using descriptions "5 longs," "3 tens" in favor of "50," "30."

Celia and Duke continued to have difficulty with tens/ones (used longs for 6 + 3, said 50 + 2 for 5 units and 2 units).

Children really need the place-value-chart for the vertical form. (Note: none was used in this lesson.)

Writing the addends (for a picture-of-manipulative display) as well as lining up digits was more than Duke could cope with.

The pictures for 10 + 1 and 10 + 2 gave Celia and Susan trouble. Teens and things with just 1 ten still are difficult.

The children (after writing a problem to correspond to a manipulative or picture-of-manipulative display) are definitely not adding symbolically in these written tasks. Even though the columns are misaligned, the sums are correct--obtained by counting blocks or counting in pictures: e.g.,

$$\begin{array}{l} 50 \\ \underline{2} \\ 52 \end{array} \quad \text{etc.}$$

Children said "40 + 50 = 90" more naturally than "4 tens + 5 tens = 9 tens" and therefore didn't relate it to 4 + 5 = 9 as readily. Showing it on abacus, Annie didn't know when to use tens and ones.

General confusion of pictures showing tens with those showing ones. Also they didn't want to say "3 tens"; said "30." I'd say "1 ten plus 6 tens equals" and every child would answer 70."

I had $\begin{array}{r}5\\+3\\\hline 8\end{array}$ and $\begin{array}{r}50\\+30\\\hline 80\end{array}$ on the board, trying to relate them. Suddenly, Omar got very excited and said, "I see, I see, there is a 5 and 3 and there is a 5 and 3. That's the same and the answer is the same." Later, Kate looked at the board and told me "both of the answers have 8," and pointed to the 8 and 80.

Initial trouble with vertical form--didn't know where to write numbers. Omar, Chris, and Paul worked with symbols only, ignored abacus. Annie used abacus for each problem, got them correct. For pictures, Annie counted beads for sum, didn't work with numerals at all.

Worksheet with numerals only--Annie, Claire, and Paul (3 of the 6 children) asked for the abacus.

Children have no trouble saying 9 or 9 ones, when giving solution preferred to say 90 rather than 9 tens. To answer 4 ones + 5 ones they'd say 9 ones, but to answer 4 tens + 2 tens they'd say 60. If I asked how many tens they could say 6 tens.

At first kids would show 60 and 50 or 20 and 60 rather than 60 and 5 or 20 and 6 (using manipulatives)--Karl in particular--for 50 and 5, he put out 10 longs. By asking him in steps (show me 50; now show me 5), he seemed to get it.

Karl, Tommy, and Calvin were confused by 5 + 1 = 6 and 50 + 10 = 60 on the abacus pictures. Working from an abacus seemed to help.

For the problem $\begin{array}{r}30\\+6\\\hline\end{array}$ on the board, Alex wrote $\begin{array}{r}30\\+6\\\hline 306\end{array}$, but he said the answer was 36. Had Alex go back and check his (manipulative) display for how many tens and ones. He wrote 36 and after some prodding said "30 + 6 = 36."

Kept Alex after (for special help) to "write" my displays. On worksheet (independent work in class) he had $\begin{array}{r}40\\+20\\\hline 06\end{array}$ $\begin{array}{r}30\\+1\\\hline 40\end{array}$ $\begin{array}{r}20\\+5\\\hline 52\end{array}$ $\begin{array}{r}01\\+2\\\hline 12\end{array}$ $\begin{array}{r}30\\+4\\\hline 40\end{array}$

They understand the pattern 2 + 5 = 7 so 20 + 50 = 70, but they have trouble with one-ten; e.g., 6 + 1 = 7 so 60 + 10 = 70.

Tommy anticipated answer to 30 + 6 would be 90 because he had just seen 30 + 60 = 90. He did this on another occasion too and was rather surprised when he counted and found he was wrong.

They have difficulty expressing answer in tens; e.g., 20 + 70 = 90, so 2 tens + 7 tens = 90 not 9 tens.

... have trouble knowing where to put things--like lining up the ones under the ones, making it difficult to see the pattern (how 2 + 6 = 8 is related to 20 + 60 = 80).

They can't tell me the answer to a problem (in symbolic form) like $\begin{array}{r} 40 \\ +30 \\ \hline \end{array}$ without objects and counting), but if I say it verbally, they can.

Kept Mary after to catch up on > and <. When she says 69 and 96 she said "These are the same. See? 9 and 9. 6 and 6 (pointing)." I asked her to read them and she realized they weren't the same and 96 was more.

Lesson 9

No notes available from one group.

David and Susan are at ease with describing blocks as "1 ten, 2 tens, ..."

Celia and Duke persist with "10, 20, ..." preference.

Duke made many errors--wrote units digits in 10's column, vice versa; wrote $\overline{20|2}$ for 22, for 4 units said there were "40...14...4."

Paul had great difficulty telling me how many tens were on the abacus. He said "30" and wrote 30 tens (for a display of 3 beads on the 10's rod). After much prodding and explanation I told him to count the beads and he said "3" and I told him to write 3, but he said it would look like ones. I showed him the word tens but he really didn't buy it.

On one problem Alex and Karl had

tens	ones
4	0

40 rather than

tens	ones
0	4

4 (the problem was to write how many tens and ones and the numeral for a picture).

Lesson 10

All pupils except Carrie could find a picture which showed the answer

to an addition problem (presented in iconic form). Carrie would focus on matching the tens but did not always match the ones, which led to mistakes.

Most of the children...had difficulty reading the intermediate step (in the addition algorithm). Phil wrote

$$\begin{array}{r} 35 \\ +21 \\ \hline 7 \\ 40 \\ \hline 47. \end{array}$$

Cubby had a lot of difficulty; he wrote for example

$$\begin{array}{r} 13 \\ +12 \\ \hline 2 \\ 25 \\ \hline 27 \end{array}$$

Carrie has a lot of difficulty working on the abstract level and lacks confidence in herself when she does not have pictures or sticks to use.

Joe Benny and Brett work extremely well at the abstract level.

To the question "Should we do tens or ones first?" Susan said immediately, "It doesn't matter." Celia and Duke count tens, 10, 20, ... others use 1 ten, 2 tens... Celia and Duke respond to "How many tens?" "40."

Form-Boards are not popular. Susan said "I hate them" and others echoed; ...for the bright ones it was a structure they didn't need.

All but Duke can describe problem in detail (40 + 30 is 70, 2 + 6 is 8 for

$$\begin{array}{r} 42 \\ +36 \\ \hline 78 \end{array}$$

Susan, David, and Washington...all began today by combining tens first, but switched on worksheets which have "ones" and "tens and ones" written beside the partial sums.

Celia used blocks on Worksheet B (symbolic problems to complete) but not on Worksheet C (pictures to problems to answer).

Washington, Susan, and David all adapted to short form easily (abbreviation of algorithm from

$$\begin{array}{r} 42 \\ +23 \\ \hline 5 \\ 60 \\ \hline 65 \end{array} \quad \text{to} \quad \begin{array}{r} 42) \\ +23 \\ \hline 65. \end{array}$$

Celia refused to use short forms. Celia has reacted this way before: the "teach me what to do and I'll do it; don't go changing it on me" kind of attitude. She was comfortable with long form and had no interest in changing.

Duke's errors: reversed digits, misplaced ones' sum in tens column, vice-versa, etc.: wrote

$$\begin{array}{r} 15 \\ +33 \\ \hline 4 \\ 80 \\ \hline 84 \end{array}$$

... I think rather than adding to get partial sum Duke counts all the units or all the longs, but doesn't put the partial sums in the right column. Whichever he does first goes in the ones column. Duke's state at this point is that he cannot cope with both setting up blocks and recording, or recording back the problem and the sum from a picture. If he has only <u>one</u> thing to do (choose a picture, write a sum) he usually can do it okay.

Verbalization of 5 tens and 3 tens gives 8 tens (for example) was impossible. Kids would say "20 and 10 is 30" but not (2 tens and 1 ten is 3 tens).

When I read problem for kids to show on abacus, Annie had trouble. If I read the second number before she had put on the first number, she got totally confused.

When Omar "read" problem from picture, he wouldn't look at picture of the answer, but would look at the abacus with addends, touch tens to put them together, then touch ones same way. It wasn't counting, but a physical and in a mental process, eyeballing it.

For pictures where I read problem, Paul chose wrong ones, sometimes reversed picture or would count (a picture for 61) as 10, 20, 30, 21, 32, 33, 34.

When I showed addends on my abacus, Annie would put down on rubber band to join tens and ones before she could find picture for answer.

Kids didn't want to write intermediate step →

$$\begin{array}{r} 23 \\ +15 \\ \hline 8 \\ 30 \\ \hline 38 \end{array}$$

Instead, they'd add ones say and write "8" then add tens, say "30" and write 3 in front of 8.

For intermediate step Kate would write $\begin{array}{r} 4 \\ \underline{84} \end{array}$ instead of $\begin{array}{r} 4 \\ \underline{80} \end{array}$.

Annie used abacus for each problem putting ones on, writing it, then putting tens on and writing it, then count all beads again and write the answer.

(To get the answer from a picture showing addends on an abacus) Paul would count tens of the first addend correctly, then start counting by ones when he got to tens of the second addend.

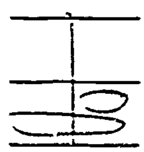
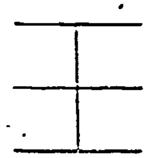
Chris and Omar said it's faster to write tens first, but they wrote ones first because I told them to. Chris added that it would slow him down, but he'd do it.

Took forever to get first example $\begin{array}{r} 13 \\ \underline{+12} \end{array}$ on the Form-Board. But they seemed to catch on by the second example.

Kids joined the ones and the tens simultaneously by columns, but kept tens and ones separated.

Karl and Alex could give me solutions to single digit problems but only Calvin and April could transfer to 30 + 40 or 40 + 50.

When addends <u>and</u> the solution are pictured kids don't want to show the problem with manipulatives because they already know the answer.

Began writing ovals  in  for Alex to write in... hopeful.

Kids seem to know how many tens and ones, but not where to write it.

I had Karl and Alex solve some problems on the board, using sticks, and pointing to what the numerals went with. Karl was very proficient; Alex somewhat slower.

They don't see why they must put the sum in the bottom panel of the addition-form-board, but do it anyway.

Strategies varied for finding the correct picture to go with the problem. Betty computed answer first, then looked for picture. Mary would count the ones then find the picture with the correct number of ones, and <u>then</u> worry about the tens.

They had difficulty deciding where to put things. They wanted to just write the answer and even when adding ones first want to do $\begin{array}{r} 24 \\ \underline{+13} \\ 37 \end{array}$

instead of 24
+13
① 7
② 30
37. They thought it strange to have only one digit on a line. They didn't use line ① and ② to find answer, just the objects. When, after they finished, I asked "So what it 24 + 13?" they were puzzled.

Neither Mary nor Tommy put middle steps in on (problems like)

2	3
+1	4

Tommy is lost without objects. Even when the problem is presented with pictures he must use objects. For example, he had difficulty counting these two (pictured) sets together: 10 10 10 |||

10 10 || He counted 10, 20, 30, 31, 32, 33, 20, 30, 32. He even had difficulty after I suggested he count the 10's first (i.e., that he count 10, 20, ..., 50, 51, 52, ..., 55).

Tommy...does not use the numerals at all and sees nothing wrong with
12
+23
5
35. He can find the answer (using objects) but sees no meaning in the symbolism.

By the end of the lesson (5 days), Tommy seemed to know what he was doing. He could add symbolically and also show you with objects. Adding multiples of ten caused Tommy and Mary problems. They can answer these problems when given orally, but the two zeros seem to throw them off at the symbolic level.

Lesson 11

Children did not have difficulty using addition-form-board.

(For some oral drill problems) All children had trouble telling whether a sentence is true or false (e.g., 5 + 3 = 8, 26 is less than 32): seemingly the words (true and false) have no meaning for them. They could tell which of two numbers is more or less by determining whether a particular statement such as "39 is greater than 59" or "27 is less than 51" is true or false.

All children except Carrie were able to select the correct pictures for a given addition problem and put them in order to show addition.

All were able to look at a picture of an addition problem with sticks and show the answer with sticks.

Carrie did surprisingly well on worksheets. I finally got her to do Worksheet F (the last) without sticks. Seemingly, she did not trust herself at the symbolic level.

Children mostly put ones together first when manipulating blocks; Susan, David, and Washington tried to group everything at once--Susan and Washington look at addends and tell answer; David physically moves blocks together first and Duke needs prompts: "How many ones, etc."

In telling about problem (recapitulating, explaining what they did with blocks, etc.) Susan and Washington are adept; David and Celia hesitate and make errors. Duke's reaction is "Aw, here we go again" --like the whole thing is too much for him.

Washington is very reluctant to use the form-board. He'll use blocks, but not on the form-board.

On symbolic worksheets, Susan and Washington begin to quit using the blocks (rely on the symbols). Washington starts leaving out the zero in the tens partial sum.

Duke, whenever I'm not right beside him, reverts to assorted errors. The most common today is to add tens and ones digits together. For example

$$\begin{array}{r} 22 \\ \underline{+5} \\ 5 \\ \underline{40} \\ 55. \end{array}$$

Celia stuck to long form, used blocks, occasionally had random errors.

$$\begin{array}{r} 25 \\ \underline{+1} \\ 6 \\ \underline{80} \\ 86 \end{array} \qquad \begin{array}{r} 4 \\ \underline{+55} \\ 9 \\ \underline{50} \\ 55 \end{array}$$

Duke, with guidance, is beginning to be able to do entire procedure. He uses blocks and lines them up all over the form board, ignoring our neatly marked spaces.

Duke and Celia (after others in group have finished) finish the symbolic worksheets, both did very well. Celia uses long form, no blocks (uses fingers). Duke uses both long and short form, reverses columns occasionally.

Given this task, "I'm going to read you an addition problem, then you find the pictures and put them in order to show the addition." Annie could not choose a picture from a group, she had to be given one picture at a time, count beads (on abacus picture) and decide if it was the one, then go to the next picture. She also can't "read" problem from picture--she has to count (slowly) then forgets what she just counted before she can say the numbers.

Annie read written problems slowly, hesitantly, but must better than telling the problems from abacus or picture.

(On symbolic worksheets) Paul had most trouble writing numerals from pictures. He did much better on completely symbolic worksheets, no pictures. Annie used abacus faithfully on every problem until I took it away on Worksheet F (the last one). She then used fingers and tapped on table, even for obvious combinations (900, etc.). Chris and Claire asked for abacus; didn't use it for any worksheets.

Karl joins the tens, then the ones; Freddy joins the ones, then the tens; Calvin varies; Susan and Alex join both together.

All abacus pictures are harder (than block or stick pictures); no trouble reading a number from a real abacus but there is from a picture.

April fills in the detail to please me: She first had

$$\begin{array}{r} 72 \\ +5 \\ \hline \\ \\ \hline 77 \end{array} \quad \text{then} \quad \begin{array}{r} 72 \\ +5 \\ \hline 7 \\ 70 \\ \hline 77. \end{array}$$

Tommy and Mary did very well with objects and okay with pictures, but the symbolic is like pulling teeth. Especially when the "middle step" is required. Both made errors like

$$\begin{array}{r} 25 \\ +4. \\ \hline 9 \\ 29 \\ \hline 29. \end{array}$$

Mary doesn't understand why she is writing the middle step, but seems to understand the answer. She never looks back at the middle step, but she uses the pictures. They don't seem to connect the short form with the long form.

Mary on 61
+7 added all the numbers together (i.e., gave 14 for answer). She told me that in problem 4
+55 the fives were the ones. She's okay using objects.

SUBTRACTION FOR NUMBERS NAMED BY 2-DIGIT NUMERALS--NO REGROUPING

The broad objective of this sequence of lessons--lessons 12-15--was to develop the subtraction algorithm for subtraction of numbers named by 2-digit numerals. This sequence of lessons was limited to subtraction without regrouping. As in the addition lessons the first work in the subtraction lessons, especially at the enactive level, gave special emphasis to review and extending the concept of subtraction as being a take-away operation; that is, the removal of a subset from a given set. The sequence of lessons was designed to present the ideas associated with the subtraction algorithm in the following order.

1. Review basic facts, minuend and subtrahend both less than 10.

2. Extend the skill with basic facts to compute differences of two numbers, both multiples of ten.

3. Computation of the difference of two numbers, both named by 2-digit numerals.

4. Computation of the difference of two numbers, one named by a 2-digit numeral, the other by a 1-digit numeral.

Lesson 12 was titled "Subtraction: Review facts, tens take away tens." The stated objectives for the enactive level were:

1. To give oral subtraction sentences which correspond to object manipulations to suggest subtraction:
   a. basic subtraction facts, sums to 9,
   b. tens and tens, sums to 90.

2. To give object manipulations for subtraction problems presented orally:
   a. basic facts, sums to 9,
   b. tens and tens, sums to 90.

Appropriate modifications of these statements to reflect iconic and symbolic activity define the objectives for the iconic and symbolic phases. In this lesson numerous problems were given together, for example 5 - 4, 5 tens take away 4 tens, 50 - 40 were presented in close succession, to help the children observe the similarity between computing 5 - 4 and 50 - 40.

Lesson 13 was titled "Subtraction: 2-digit and 2-digit without regrouping." In this lesson a Subtraction-Form-Board was introduced. The purpose of the form-board, as for addition, was to provide a structure on

which the children would display objects for the "beginning set," then remove a subset and keep it displayed in a designated spot, and finally move the remainder set to a designated spot. The subtraction-form-boards were constructed from light cardboard about the size of an open file folder.

At the iconic level of instruction a subtraction problem was suggested in a picture by ringing the "removed-set" and extending the ring to an arrow to suggest removal. The stated objectives for the three phases of instruction for lesson 13 were as follows.

Enactive level

1. To find the answer to 2-digit and 2-digit subtraction problems, without regrouping, using the objects.

Iconic level

1. Given a picture-of-objects display of subtraction with two 2-digit numbers, to show the same with blocks and find the answer.

2. When given a subtraction problem with two 2-digit numbers orally, to choose the picture which shows the subtraction and give the answer orally.

3. Given a block display of a subtraction problem with two 2-digit numbers, to find the picture which shows the subtraction and give the answer orally.

4. Given a picture-of-objects display of a subtraction problem with two 2-digit numbers to.
   a. tell what subtraction problem is shown,
   b. display the answer with objects.

Symbolic level

1. When shown a subtraction problem with two 2-digit numbers in symbolic form to
   a. read the problem,
   b. show and solve the problem using objects, and give the answer orally,
   c. solve the problem with objects and write the answer,
   d. solve the problem by thinking about subtracting in the ones and tens place and write the answer,
   e. after having completed a problem and solution with blocks be able to point out corresponding numerals and objects; e.g., in 32, 2 goes with 2 units, etc.

2. When given a subtraction problem with two 2-digit numbers in oral, iconic, or enactive mode, to
   a. symbolize the problem,
   b. solve the problem with objects,
   c. solve the problem from the picture by counting,
   d. write the answer.

During the first part of the symbolic level of lesson 13, use of a place-

value-chart was emphasized. Later in this lesson, illustrations and worksheets were presented without the place-value-chart. The place-value-chart was used initially to provide a structure for the algorithm work and was gradually removed.

The form of the subtraction algorithm which was taught is illustrated below.

$$\begin{array}{r}39\\ \underline{-15}\\ \end{array} \rightarrow \begin{array}{r}39\\ \underline{-15}\\ 4\end{array} \rightarrow \begin{array}{r}39\\ \underline{-15}\\ 24\end{array}$$

It was decided not to use an expanded algorithm for subtraction, such as

$$\begin{array}{r}39\\ \underline{-15}\\ \end{array} \rightarrow \begin{array}{r}39\\ \underline{-15}\\ 4\end{array} \rightarrow \begin{array}{r}39\\ \underline{-15}\\ 4\\ \underline{20}\end{array} \rightarrow \begin{array}{r}39\\ \underline{-15}\\ 4\\ \underline{20}\\ 24\end{array}$$

However, it was observed that some children developed such an algorithm on their own. This obviously was a transfer from the form used for the addition algorithm. Such an expanded algorithm for subtraction appears to be feasible.

Lesson 14 was titled "Subtraction: 2-digit and 1-digit and 2-digit and tens without regrouing." The stated objectives of lesson 14 for the enactive, iconic, and symbolic levels were as follows.

<u>Enactive level</u>

1. To give oral subtraction sentences which correspond to object manipulations to suggest subtraction--2-digit minus 1-digit numbers and 2-digit minus tens.

2. To give block manipulations for subtraction sentences which are given orally.

<u>Iconic level</u>

1. Given a picture-of-objects display of subtraction of 2-digit minus 1-digit or 2-digit minus tens to show the same with blocks, find the answer, and orally state the subtraction problem as a complete sentence.

2. When given such a subtraction problem orally, to choose the picture to show the subtraction, give the answer orally, and state orally the problem and answer as a complete sentence.

3. Given a picture-of-objects display of such a subtraction problem to
   a. tell orally the subtraction problem shown,
   b. display the problems with objects,
   c. state orally the subtraction problem and answer as a complete sentence.

Lesson 15, the final lesson for the sequences of lessons on addition and subtraction without regrouping, was titled "Addition/Subtraction Review."

It was presented at the symbolic level only. It was the only lesson in the sequence in which addition and subtraction problems were presented to children in the same teaching session and on the same worksheets. Thus it was the first lesson in which children had to, on each problem determine whether the problem was addition or subtraction; this was a nontrivial task for some children. The stated objectives for this lesson were as follows.

1. Given addition and subtraction sentences in oral mode to write the sentence in vertical form.

2. Given addition and subtraction problems in oral mode, to
   a. write the problem,
   b. solve the problem,
   c. read the problem and answer as a complete sentence.

3. Given addition and subtraction problems, without regrouping, in the iconic mode, to
   a. write the problem and the answer,
   b. read the problem and answer as a complete sentence.

4. Given addition and subtraction problems in symbolic form, to
   a. find the answer,
   b. read the problem and the answer as a complete sentence.

LOG EXCERPTS

Lesson 12

On the iconic level, Brett, Cubby, Opie, and Carrie found the picture for 5 tens take away 2 tens when they should have found the picture for 5 ones take away 2 ones.

On a worksheet (with pictures for both addition and subtraction), for the picture of 2 bundles and 4 sticks plus 2 sticks, Carrie wrote $\begin{array}{r} 204 \\ \underline{+11} \end{array}$.

Carrie has to be taught how to record so ones will be under ones, tens under tens, etc. Example (she wrote) $\begin{array}{r} 60 \\ \underline{-40} \\ 20 \end{array}$.

Children confused pictures depicting addition and those depicting subtraction.

This is a particularly effective and smooth transition to symbolic work. All the children were competent at showing with blocks the problems which I wrote on the board. All but Duke wrote $\begin{array}{r} 9 \\ \underline{-8} \end{array}$ for $\begin{array}{r} 9 \\ \underline{-1} \end{array}$

and $\begin{array}{r} {}^{7}20^{0} \\ \underline{-50} \end{array}$ for $\begin{array}{r} 70 \\ \underline{-20} \\ 50. \end{array}$

In group work, Duke does procedures with blocks okay and records well except for column alignment. Susan and Celia have gone through this same stage--when they were recording a problem that they've gotten from a picture, they didn't at first focus on the columns, just on three numbers. Duke is now writing $\begin{array}{r} 60 \\ \underline{-40} \\ 20. \end{array}$

Annie and Kate would sometimes say 5 - 3 instead of 8 - 3, looking at beads at the bottom. As minuend instead of answer. Kate, when trying to tell me statement in "tens", would say "5 tens take away 30 tens equals 20."

I think all see the relationship between 8 - 3 and 8 tens - 3 tens in oral drill but only Calvin and April carry it over to 80 - 30, but all seem to know the same display works for 80 - 30 as 8 tens - 3 tens.

Had $\begin{array}{r} 90 \\ \underline{-20} \end{array}$ for Calvin to show. Alex said it was easy because $\begin{array}{r} 9 \\ \underline{-2} \end{array}$ was still on the board.

On Worksheet A (corresponding to picture depicting 50 - 30), Alex wrote $\begin{array}{r} 30 \\ \underline{-20} \\ 20 \end{array}$ for $\begin{array}{r} 50 \\ \underline{-30} \\ 20. \end{array}$

Alex talks to himself as he works: $\begin{array}{r} 80 \\ \underline{-50} \end{array}$ "Zero take away zero...zero; 8 take away 5...3." Karl says "80 take away 50."

Alex writes for (a display of $\begin{array}{r} 24) \\ \underline{+2)} \end{array}$ $\begin{array}{r} 204 \\ \underline{2} \\ 206. \end{array}$

Tommy and Mary interpreted a picture like as 6 - 2 = 6. They don't know their facts so this presents no contradiction.

When I showed a (subtraction) problem with objects and they wrote the problem, I was surprised. Tommy did well right off, although a similar task with addition caused him a great deal of difficulty. Abel, who breezed through addition couldn't do it. For a display for 6 - 2 = 4,

he wrote 624. After I showed him on the board he still couldn't put the numbers in the right places. For 60 - 20 = 40, he wrote -602040.

The first problem 60 - 60, which I showed on the abacus confused them all, even Betty. Tommy wrote

$$\begin{array}{r} 60 \\ -0 \\ \hline 60 \end{array}$$

Abel and Mary were lost.

On the first problem of the worksheet which had pictures depicting addition and subtraction, Betty did this. 

$$\begin{array}{r} 24 \\ 2 \\ \hline 80 \end{array}$$

Lesson 13

No problems were observed in learning to use the subtraction-form-board.

Carrie often records a ones' digit in the tens' place when writing a problem that I read or one that she wrotes on her own based on pictures.

Example:  Carrie wrote:

$$\begin{array}{r} 17 \\ -4 \\ \hline 13 \end{array}$$

Recording problems seemed to be the main difficulty of this lesson (iconic level).

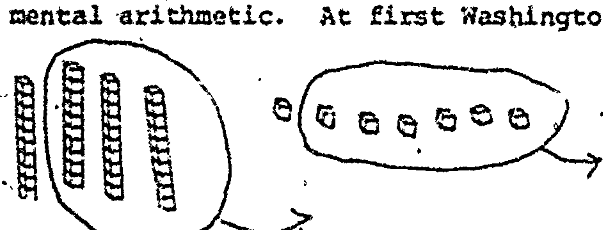
All students (during the symbolic level) seemed to tire of using the subtraction-form-board. All seemed to work correctly without the form-board except Carrie.

Once again, all five children function well at the enactive level. They were able to recapitulate the problem.

At iconic level, everyone had to count pictures carefully, to get differences as well as subtrahend and minuend. Some of the fluency in telling the problem seems lost when working from pictures.

All five children had trouble with the last item (I show picture for subtraction, they show answer with blocks). Susan and David caught on to just getting out blocks for the part not cirtled.

Worksheets which had pictures depicting subtraction. Children wrote corresponding problem and found answer: Susan and Washington get answers by counting blocks in picture. David seems to rely more on mental arithmetic. At first Washington did some funny things:



$$\begin{array}{r} 47 \\ -36 \\ \hline 2 \end{array}$$

Duke has discovered that he can do subtraction problems column by column on his fingers. He does worksheets today mostly without blocks.

On one worksheet (3) problems were presented in pairs as

3	8	38
-2	4	-24

to make a transition away from the place-value-chart. Duke did these three problems by columns (tens first, of course). He didn't even recognize that he was doing the same problem twice.

Annie has difficulty showing a number on abacus when she hears it, but doesn't see it written. I'd sometimes have to touch the tens rod and say "30," then touch the ones rod and say "7" for 37.

At the iconic level: Annie couldn't tell how many were taken away. Paul did better, but he would reverse the take-away number. He found 47 - 13 instead of 47 - 31. Claire and Kate reversed digits on the minuend (found 45 - 14 instead of 54 - 14). However, when I displayed a pictured problem on a real abacus, everyone did well.

On symbolic worksheets, Paul counted beads in picture for each number; he ignored symbols. Annie sometimes forgot which beads to count for answer. Kate had trouble with pictures too, but did better with each worksheet.

Chris has realized it isn't necessary to put zero in tens column, wrote 02, then erased 0.

Alex waits to see what to write (the teacher wrote answer on chalk board after children were to write) from the board, even though he uses the manipulative correctly. April and Calvin first write the solution to a problem, then show the display.

Kids could relate numbers (on a worksheet that depicted a subtraction situation with pictures) with the pictures.

Karl's counting backwards for subtraction and is usually off by one.

Tommy had trouble reading 47 - 31 on an abacus picture. He said "40 - 1, 47 - 30, 7 - 1, 47 - 1 ...," but did well on the other pictures. It is amazing how his speed and accuracy decrease as we progress through the three levels.

On a worksheet that has pictures to depict subtraction and the associated symbolic problem, Betty does them without using the pictures; others, especially Tommy, are dependent on them.

On a worksheet children were given oral directions as follows: Were read a number, told to show the number with objects, told what number to take away, told to take away objects, told to write the number taken away, ... Having them take away before they write how many they are taking away may not be too good or natural. They prefer to write, take away, then write the answer.

With symbolic exercises, Tommy had difficulty when there weren't any pictures. He started adding subtrahend and minuend together, subtracting the minuend, and getting the subtrahend.

Lesson 14

Carrie and Cubby made numerous mistakes; instead of subtracting to get answer they simply brought down the minuend or subtrahend. Example:

$$\begin{array}{r} 24 \\ -20 \\ \hline 20 \end{array} \qquad \begin{array}{r} 24 \\ -2 \\ \hline 24 \end{array}$$

David had a lot of trouble with the worksheet on which he had to ring the subtrahend blocks--he circled the difference, or one digit of the difference, or part difference and part subtrahend (for 56 - 44, he circled 1 long and 4 units, the exercise presented a picture of 5 longs and 6 units and the problem $\begin{array}{r} 56 \\ \underline{-44} \end{array}$, children were asked to indicate the problem on the picture).

Celia seems to work from symbolic form, even with pictures. She works in column, writes unneeded 0's, doesn't like worksheets where she doesn't get to write the problem down, sometimes ignores pictures and adds the two numbers when the complete symbolic problem (and answer) is already given.

Duke worked happily with blocks on first worksheets without pictures. This is perhaps the first time the pictures have meant enough to him to want to use blocks after when the pictures stop.

No comments from one group.

On a task where teacher read a subtraction problem and children were to show the problem with objects: Alex wanted me to write the problem on the board before he made his display. He may be very proficient at reading how many tens and ones, but not at reading the whole number.

Karl had errors such as $\begin{array}{r} 87 \\ -5 \\ \hline 52 \end{array}$ or $\begin{array}{r} 32 \\ -12 \\ \hline 22 \end{array}$

On a worksheet where children were to complete a picture (i.e., ring the subtrahend) to show subtraction: Betty would draw a ring to show subtraction of ones, write the result, then do the tens.

Tommy tried hard to convince me that 8 - 6 = 7 when, in fact, he had forgotten to take away the other 6 fingers to make 6. His number sense is poor. He needs objects but won't use them with the worksheets.

Most frequent error (when writing a problem to go with a picture depicting subtraction is

$$\begin{array}{r} 38 \\ 2 \\ \hline 18 \end{array} \quad \text{, for} \quad \begin{array}{r} 38 \\ 20 \\ \hline 18 \end{array}$$

When asked to read the problem they usually notice the error.

Tommy relies completely on pictures when they exist, as does Mary.

Lesson 15

No comments from one group.

Susan, David, and Washington have good command of both addition and subtraction algorithms. They can demonstrate and explain problems.

Paul and Annie had a hard time writing teen numbers.

Alex. wrote 15 for 51 and 50 for 15. He seems to have the most trouble with 30's and 50's.

All children had trouble with picture-of-abacus addition problem. They did okay on a picture-of-abacus subtraction problem.

Karl's errors seemed to be adding or subtracting when he should have subtracted or added, or basic fact errors due to counting back

$$\left(\begin{array}{r} 80 \\ -50 \\ \hline 40 \end{array} \quad \begin{array}{r} 48 \\ -25 \\ \hline 24 \end{array} \right).$$

Tommy does iconic and symbolic worksheets differently. He relies completely on pictures, when they are present, ignoring the numbers. On symbolic worksheets he relies on his fingers (he doesn't use manipulative objects).

ADDITION OF NUMBERS NAMED BY 2-DIGIT NUMERALS--WITH REGROUPING

The broad objective of this sequence of lessons--lessons 16-18--was to extend the children's concept of addition to addition of numbers named by 2-digit numerals in which regrouping from the units to the tens was involved. The sequence of lessons began with a rather long lesson on the "regrouping" process; that is, changing a manipulative aid or symbolic representation of a given number to have more or fewer tens or ones. The regrouping activity was then incorporated into the lessons on addition.

Lesson 16 was titled "Regrouping and Renaming." The stated objectives for the enactive level of the lesson were:

1. When shown an object demonstration for 1 ten and 10 ones, to
   a. say that they represent 1 ten and 10 ones, repectively,
   b. tell that each represents the same number,
   c. tell that each may be replaced by the other.

2. Shown a block demonstration such as 2 tens and 3 ones and 1 ten and 13 ones, to
   a. say each in words,
   b. tell the number of each,
   c. tell that each has the same number.

3. When presented a phrase like "3 tens and 4 ones" orally, to
   a. show it with objects,
   b. regroup it to represent 2 tens and 14 ones.

4. When presented a phrase like "2 tens and 15 ones" orally, to
   a. show it with objects,
   b. trade 10 ones for 1 ten,
   c. tell how many there are in all.

The objectives for the iconic level were as follows:

1. Given a picture of objects which displays a regrouping, to
   a. orally explain the steps in the regrouping,
   b. show the regrouping with objects.

2. Given a number such as 16 ones or 2 tens and 7 ones, to
   a. orally explain how to trade tens and ones to make more or fewer ones,
   b. find the picture which corresponds to the result of the trading.

The objectives for the symbolic level were as follows:

1. When given a number such as 43 in symbolic mode as "43" or "4 tens and 3 ones" or $\overline{\;3 \mid 13\;}$, to

   a. read the number,
   b. show the number with objects,

c. trade to make more or fewer ones,
d. use symbolism as in the subtraction algorithm to record the trading.

2. When shown a step-by-step written recording of a regrouping to make more ones, to
   a. show this with objects,
   b. orally explain the regrouping--from the symbols and from the blocks.

3. When shown a trading with objects or with a picture of objects, to
   a. orally explain the regrouping,
   b. use symbols to record the regrouping.

An example of a task which children were required to perform at the symbolic level was to show with objects the regrouping suggested by

	15
	~~5~~

Similarly, children were shown a regrouping with objects and asked to record the regrouping symbolically.

Lesson 17 was titled "Addition: 2-Digit Addends With Regrouping." The objectives for the three phases of instruction were as follows:

Enactive phase

1. Given an addition problem orally, to
   a. exhibit the addends with objects,
   b. perform the addition--join ones and join tens,
   c. do the necessary trading,
   d. tell the problem which was worked--i.e., read the problem in the form "addend plus addend equals sum."

2. When shown an addition problem with blocks, to
   a. tell the addends and the sum,
   b. verbalize the steps in the addition--i.e., join the ones, trade to make tens and ones, etc.

Iconic phase

1. Given a picture-of-objects display of an addition problem with regrouping, to
   a. show the problem with objects,
   b. orally give the addends and answer of the problem,
   c. orally explain the regrouping.

2. Given an oral statement of an addition problem with regrouping, to
   a. find the picture which shows the problem,
   b. orally give the addends and the answer of the problem,
   c. orally explain the regrouping.

Symbolic phase

1. When shown an object or a picture-of-objects display of an addition problem with two 2-digit addends with regrouping, to
   a. show the problem with objects,

b. solve the problem using objects,
c. solve the problem mentally,
d. record the steps in symbolic form to show joining ones, trading, and joining tens,
e. read the problem and the answer.

The algorithm that was taught to the children and on which support was provided during the enactive phase of instruction was as illustrated in the problem

$$\begin{array}{r} 35 \\ +27 \\ \hline 12 \\ \underline{50} \\ 62 \end{array}$$

A modified Addition-Form-Board was used in the lessons. Many of the worksheets which involved symbolic work had a form for children to use in the algorithm work as illustrated below:

$$\begin{array}{r} 29 \\ +37 \\ \hline \end{array}$$

ones in all
___ tens in all

altogether

This symbolic format coincided with the format of the Addition-Form-Board. It was hoped that this correspondence in style would facilitate children's symbolic recording of the algorithmic process.

Lesson 18 was entitled "Addition: 2-digit and 1-digit with regrouping." Because of our observations with the comparable sequence with addition without regrouping we decided that this lesson could be considerably shortened. Our observation was the 2-digit and 1-digit addition came very easily to children after having had 2-digit and 2-digit addition. Therefore, lesson 18 was presented at the symbolic level only. The stated objectives for lesson 18 were essentially the same as for lesson 17.

Before starting lesson 16 it was discovered that many of the children were not proficient with basic addition and subtraction facts which involved sums greater than nine. Therefore, a number of days were spent giving instruction and practice on basic facts. The children were taught a count-on strategy for addition and both count-on and count-back strategies for subtraction.

LOG EXCERPTS

<u>Lesson 16</u>

Cubby and Carrie had difficulty understanding that the same number could be expressed in two different ways. If you asked them if 12 ones were the same as 1 bundle and 2 sticks they would say "no" or act unsure about their answer.

All children had trouble with the question, "Are there less than 10 ones?" So I rephrased the question and asked, "Do you have enough to make a bundle?" All children except Carrie caught on to the trading concept for more or fewer ones. However, Cubby is not always sure that the number of sticks in all remains the same.

Children are beginning to verbalize more freely about what happens when they trade for more or fewer ones and seem to understand the written form for representing the trading process.

Cubby finally udnerstands that the total number of sticks remains the same after trading.

All children except Joe Benny made some errors on Worksheet D (a worksheet that required symbolic recording of regrouping). Some sample errors:

Phil: $\begin{array}{c|c} 5 & 10 \\ \cancel{6} & \cancel{5} \end{array}$, $\begin{array}{c|c} 1 & 11 \\ \cancel{2} & \cancel{0} \end{array}$ Opie: $\begin{array}{c|c} 7 & 18 \\ \cancel{8} & \cancel{9} \end{array}$,

Carrie: $\begin{array}{c|c} 8 & 5 \\ \cancel{9} & \cancel{5} \end{array}$

After the first example of counting after a trade, Susan and David no longer count to determine how many altogether, they conserve and remember how many there were before the trade. Celia counts the "new blocks," if she makes an error in trading or counting she trusts the new number and assumes that the amount has changed.

Celia had trouble with the question, "Are there more/less than 10 ones?"

Question "Are there less than 10 ones?" was not consistently answered correctly by anyone except Chris.

Trading a ten for more ones seemed easier than trading ones for a ten.

Omar would pick up 10 ones and turn them over (from the back to the front of the abacus), not bringing over the ten as exchange.

I changed the question to "Do we have enough ones to trade?"

An exercise with pictures showed, for example, 4 tens and 11 ones on abacus. Kids were told that I had a picture which showed the sticks after a trade. They were to guess the trade and show the results with their abacus. This was very difficult for the kids, impossible for Annie and Kate. Kids couldn't construct in their minds what they couldn't see.

In response to a symbolically recorded trading: Claire and Chris gave fluent verbal description, Omar and Paul did okay, Kate was hesitant, Annie couldn't verbalize at all.

Kids readily accepted that 12 ones or 1 ten and 2 ones were the same value but on amounts such as 35, 87, or 77, where it ended up with 7 tens and 17 ones, children would have to count again to say 87. That is, 8 tens and 7 ones: kids say this is eighty-seven; trade 1 ten for 10 ones, 7 tens and 17 ones, all agree that the same value but must count to find 87; but not the case for 12 where not so numerous ones.

...kids just seem to see it as a trading game without any purpose.

On something like 8 | 15
~~9~~ | ~~5~~ kids are explaining by using words "take away," rather than "trading." For example, Alex suggested you were "taking away a long for 10 units."

Karl and I worked on symbolic problems like 6 | 18
~~7~~ | ~~8~~ at the board with displays. Karl caught on and redid his worksheet. Alex is okay with trading on the display and talking about it but not with written symbols.

Trading came easy, but they weren't always sure, at first, that they had the same number.

They are (now) convinced that the total number does not change with trading. Have difficulty deciding whether to exchange 1 ten for 10 ones or vice versa.

Showing what the picture would look like after trading 10 ones for 1 ten was not too difficult. Going the other way didn't make much sense ...they see no need as yet.

No difficulty with trading but recording is a problem. Betty started doing 8 | 4
~~9~~ | ~~5~~ --corrected herself eventually.

Mary still recounts everything after trading, the others knew the number had been preserved.

Lesson 17

Addition-form-board was introduced today but all children had difficulty displaying the correct number of sticks in the "tens in all" line. They wanted to include the ten they got as a result of trading the ones to tens and ones.

Phil made mistakes because he counted the ten he traded twice. Example:

$$\begin{array}{r} 38 \\ +57 \\ \hline \end{array}$$

15 ones in all
90 tens in all

95 altogether --correct answer, but he apparently got it by counting the picture correctly.

Cubby made errors on the "tens in all" lines. Examples:

$$\begin{array}{r} 46 \\ +39 \\ \hline 15 \\ \underline{46} \end{array} \qquad \begin{array}{r} 84 \\ +15 \\ \hline 19 \\ \underline{80} \end{array} \qquad \begin{array}{r} 69 \\ +27 \\ \hline 16 \\ \underline{30} \end{array}$$

Celia and Duke combine the ten from trading ones with the te.s from the addends in the "tens in all" space quite frequently.

Duke is unable to work along. With pictures he must match with blocks and combine, trade, etc.

Celia needs guidance only to remind her of the next step.

Susan counted the ones in all for first several problems, then added ones in symbolic form, without checking against picture. Washington went through the same progression.

Susan and Washington work without blocks, accurately in symbolic form.

Free form (i.e., as opposed to structured) place-value-chart was messy for all children.

Duke finished (last symbolic worksheet) needing blocks and my help until last six problems. Got five out of last six right using blocks.

They forgot to trade for a ten, would just throw over the 10 ones, ended up ten short.

When writing "ones in all" after trading for a ten on abacus, kids didn't want to count in the ten bead as part of "ones in all."

Chris and Omar ignored pictures, worked only symbolically.

With the abacus (addition with regrouping), I think Calvin and April follow what's going on. Karl can do it but I'm not sure he knows what he's doing. Alex is following each direction with no idea of a total product.

Alex is getting frustrated with the abacus. He easily places the initial display, adds the ones, then hesitates and watches the others before he considers the need to trade.

Alex had six errors--four of them due writing backwards; for example:

$$\begin{array}{r} 15 \\ +67 \\ \hline 12 \\ 07 \\ \hline 19. \end{array}$$

Alex told me it was hard to write numbers.

They prefer to trade and put everything together in one big step--getting the right answer is all that is important.

Lesson 18

Most of the children did well.

Paul and Claire did subtraction in "long" form like addition.

Karl and Alex not real confident. Karl mixing addition and subtraction, such as:

$$\begin{array}{r} 82 \\ -22 \\ \hline 4 \\ 60 \\ \hline 64. \end{array}$$

SUBTRACTION OF NUMBERS NAMED BY 2-DIGIT NUMERALS--WITH REGROUPING

The broad objective of this sequence of lessons--lessons 19 and 20--was to extend the children's concept of subtraction to subtraction of numbers named by 2-digit numerals in which regrouping was required. Lesson 16 had introduced the children to the regrouping processes needed for subtraction. Moreover, this skill was reviewed as part of drill activities in the addition lessons. The algorithm that was taught in these lessons is illustrated with the problem

$$\begin{array}{r} 3\;13 \\ \cancel{4}\cancel{3} \\ -19 \\ \hline 24. \end{array}$$

During the enactive phase of each lesson foundation was laid for this symbolic algorithm.

Lesson 19 was titled "Subtraction: 2-digit minus 2-digit with regrouping." The stated objectives for the enactive phase are as follows:

1. Given a problem like 5 - 9, to use blocks, count-back strategy, and count-on strategy to explain that this is not possible in the numbers we know.

2. Given a problem like 35
   -19, to exhibit the minuend with objects and to explain with the objects that we can't take 9 ones away from 5 ones.

3. Given a subtraction like 35 - 19 in oral form, to
   a. exhibit the minuend with objects,
   b. do the necessary trading,
   c. "take away" the appropriate number of ones and tens,
   d. tell the answer of the subtraction,
   e. say the problem in the form "35 take away 19 equals 16 "

4. When shown a subtraction problem with objects, to
   a. tell the minuend, subtrahend, and remainder,
   b. verbalize the steps in the subtraction; e.g., trade 1 ten for 10 ones, add the 10 ones to the ones, remove ones, remove tens.

Appropriate modifications of these statements to represent iconic and symbolic activity defines the objectives for the iconic and symbolic instructional phases.

Lesson 20 was titled "Subtraction: 2-digit and 1-digit with regrouping." This lesson was presented at the symbolic level only. The stated objectives for this lesson are very similar to those for lesson 19. In lesson 20 quite a number of worksheets were given to the children which involved all the different kinds of addition and subtraction problems they had learned--that is, subtraction and addition on the same page so that decisions about whether to add or subtract had to be made; subtraction and addition problems with regrouping and without regrouping so decisions about whether or not to regroup were necessary.

LOG EXCERPTS

Lesson 19

Cubby and Carrie have trouble deciding which of two numbers is greater when it is necessary to trade.

Phil knows how to trade 1 ten for 10 ones but sometimes forgets to remove his ten. Others understand when and how to trade and had no difficulties.

Cubby and Carrie continue to have trouble deciding which of two numbers is greater and when it is necessary to trade.

Subtracting smaller numbers from larger numbers instead of trading occurs from time to time on subtraction lessons. Some errors noted:

$$\begin{array}{r} 9 \\ \cancel{4}8 \\ \underline{-39} \\ 61 \end{array} \qquad 69 - 17 \text{ was recorded as: } \begin{array}{r|l} 5 & 2 \\ 6 & 9 \\ \hline -1 & 7 \end{array} \qquad \begin{array}{c|c} 2 & 13 \\ \cancel{1} & \cancel{3} \end{array}$$

$$\begin{array}{c|c} 3 & 10 \\ \cancel{4} & \cancel{1} \end{array} \qquad \begin{array}{r} 13 \\ \underline{-5} \\ 0 \\ \underline{00} \\ 00 \end{array} \qquad \begin{array}{r} 2\ 16 \\ \cancel{3}\ \cancel{6} \\ \underline{-2\ 3} \end{array} \qquad \begin{array}{r} 2\ 17 \\ \cancel{3}\ 0 \\ \underline{-1\ 7} \\ 1\ 0 \end{array} \qquad \begin{array}{r} 4\ 8 \\ \cancel{5}\ \cancel{8} \\ \underline{-3\ 9} \\ 1\ 1 \end{array}$$

Cubby had difficulties trading and showing what's left symbolically, trades when unnecessary, doesn't seem to know how to determine when he doesn't have enough ones.

Susan suggested trading in order to subtract 37 from 45. All the children tried it--Susan did it perfectly, Celia traded and wound up with 14 units, Duke traded and would up with 10 units. David and Washington made correct trades but subtracted wrong.

Duke at one point had 2 longs and 4 longs out and said excitedly, "Look, Stewart, 2 + 4...I know how to show 2 + 4 = 6!"

Emphasized that total after trade should be same as total before--Celia's trading improved.

Lots of resistance to blocks today. At first, the children felt they "already knew how to do it." The step-for-step matching of blocks to symbolism became a game for them.

After finishing problem, when I asked "what does this (50) mean?" (in $\begin{array}{r} 50 \\ \underline{-27} \\ 23 \end{array}$) David put all his blocks back together and said, "it's all the blocks." Even in subtraction he thinks total rather than parts.

At first Susan misjudged when trade was needed. When told she made an error she corrected it. She did tens first and "subtracted up." By second worksheet, she went from picture to writing problem, worked problem symbolically. She said she knew when to trade "from the ones" and worked accurately.

Duke still has to count ones after trading--doesn't see "ten-more" pattern.

Susan traded all the time, worked symbolically, wrote: $\begin{array}{r} 6\ 10 \\ \cancel{7}\ \cancel{0} \\ \underline{-4\ 0} \\ 3\ 0 \end{array}$. Like David

the other day she ignores a trade she has written when she decides she doesn't need it. She has lost her command of which ones digit is greater as a cue.

Duke used blocks and worked through the procedure accurately. Would say confidently (after displaying minuend), "gotta trade" or "I can do it."

Celia used blocks, explained well, worked slowly.

David, Susan, and Washington quite competent. By end of day, Celia was working problems accurately without blocks. Duke still needs them.

Kids immediately rejected "4 take away 8" types--said "you don't have enough." When 45 - 17 was presented, Paul said, "It's just impossible! You need more ones." Chris immediately suggested trading.

Claire began trading a ten for the number of ones she needed to subtract instead of ten ones.

Kids were very excited to (re-)discover that trading did not change the value.

Kids couldn't work on abacus and write step-by-step. They either worked problem through on abacus and then wrote, or else worked problem through on paper, then showed with abacus.

For 45 - 17, April immediately thought to trade; Calvin and Karl eventually traded; Alex didn't until I suggested it and we did it together.

Calvin and Karl sometimes get the wrong answer due to not returning the ten after trading.

Alex seems to be counting the pictures and writing but has no feel for what he's doing and doesn't get a feel for a problem and a solution.

Alex has much difficulty with symbolic work. In recording trading he got:

$$\begin{array}{r} 15 \\ 4\cancel{5} \\ \underline{-19} \end{array}$$

6 --okay, but I'd have to remind him to show how he traded away one ten. He sees no relationship between these problems and some drill activities which deal with regrouping only:

$$\begin{array}{r|l} 2 & 17 \\ \cancel{3} & 7 \end{array}$$

They realized you couldn't do 4 - 9, but also thought, at first, that 45 - 17 couldn't be done either. Betty and Abel saw that trading was necessary. It took Tommy a long time...he isn't clear about what trading means.

Most common errors are: putting 10 ones instead of correct number after trading; trading when it isn't necessary.

Mary needs objects for everything. Abel uses them occasionally, usually to correct an error.

Neither Tommy nor Abel was good at telling me why they do what they do.

Lesson 20

Joe Benny, Opie, and Carrie had difficulties deciding on which operation to do (addition or subtraction). At times they had difficulty translating pictures into symbols. Phil had a good grasp of what he was doing and did a beautiful job.

Phil had difficulty with problems that had 0 ones in minuend and would subtract 0 ones in minuend from ones in subtrahend. Example:

$$\begin{array}{r} 30 \\ \underline{-15} \\ 25. \end{array}$$

Most children had no difficulty.

Duke used blocks all period. Counted blocks (2 longs and 6 units) as 10, 20, 30, ..., 80.

Kids did well with mix (addition and subtraction) of problems, could tell when regrouping was necessary.

Paul showed subtraction in two steps like addition:

$$\begin{array}{r} 64 \\ \underline{-24} \\ 0 \\ \underline{40} \\ 40. \end{array}$$

When Annie saw that trading was needed, she'd hold up all ten fingers, look at them a few seconds, then take away ones in subtrahend and write that as answer. I told her to trade and write number of ones including those in minuend. She still help up ten fingers each time before she could write the trading numerals; it's like a visual association with abacus trading.

On symbolic problems, kids seem to know if they must trade, but Karl and Alex don't always return the ten.

Worksheet with mixed + and - problems hard for all kids. Kind of a mind set.

3-DIGIT NUMERATION

The final lessons--lessons 21 and 22--for the teaching experiment were directed toward the broad objective of extending children's place value number concepts to numbers named by 3-digit numerals. Special emphasis was given to counting activities, displaying "large numbers" with objects, and when given a representative of a number either in enactive, iconic, or symbolic mode to tell the number and to display and tell numbers which were 1, 2, 10, and 20 more (and less).

Lesson 21 was titled "3-digit numerals: counting to one hundred by ones, counting 100-125 by ones, counting 100-200 by tens." The stated objectives for the three phases of instruction were as follows:

<u>Enactive phase</u>

1. Children will investigate the concept of 100 and will be able to
   a. display 100 ones and trade for 1 hundred,
   b. display 10 tens and trade for 1 hundred,
   c. display 1 hundred and trade for 100 ones,
   d. display 1 hundred and trade for 10 tens.

2. When read a number like 53, to
   a. display the number with objects,
   b. display and tell the numbers which are 1, 2, 10, and 20 more,
   c. display and tell the numbers which are 1, 2, 10, and 20 less.

3. Display objects to show counting from 100 to 125.

4. Display objects to show counting from 100-200 by tens.

5. Count object displays for numbers like 130 as 10, 20, 30, ..., 130, and as 100, 110, 120, 130.

6. Group tens to hundreds and tens.

7. When given x tens ($11 \leq x < 20$) to display these as 1 hundred and $(x - 10)$ tens and tell that x tens equal 1 hundred $(x - 10)$ tens.

8. When read a number like 150, to
   a. display the number,
   b. display and tell the numbers which are 1, 2, 10, and 20 more,
   c. display and tell the numbers which are 1, 2, 10, and 20 less.

<u>Iconic phase</u>

1. Children will investigate the concept of 100 and will be able to

a. recognize a picture with ten sets of 10 ones as 100,
b. tell that 100 ones is equal to 10 tens.

2. When shown an object display for numbers like 53, to
   a. tell the number which is shown,
   b. find the picture which shows numbers which are 1, 2, 10, and 20 more (and less).

3. Investigate counting from 100 to 125.

4. Arrange picture sets to show counting from 100 to 200 by tens.

5. Count pictures of block displays for numbers like 150 as 10, 20, ..., 150 and as 100, 110, 120, ..., 150.

6. When read a number such as 160, to find pictures of block displays which show this as 16 tens and as 1 hundred and 6 tens.

7. When shown a picture-of-objects for 1 hundred and 5 tens, to display this with tens objects and tell that 150 is 15 tens.

8. When shown a picture-of-objects for 15 tens, to display the tens with objects as 1 hundred and 5 tens and tell that 15 tens equal 150.

9. To extend the concepts of more and less to numbers less than 1000.

Symbolic phase

1. When given a number like 53 in symbolic form, to
   a. display the number with objects,
   b. display and write the numbers which are 1, 2, 10, and 20 more (and less).

2. When shown an object or picture-of-objects display, which shows counting from 100 to 125, to write the corresponding numerals.

3. To write the numerals in sequence from 100 to 125 without enactive or iconic displays.

4. When given an object or picture-of-objects display which shows counting from 100 to 200 by tens, to write the corresponding numbers.

5. When given an object or picture-of-objects display for numbers like 130, to write the number.

6. When given a number like 130 in symbolic form, to show the number with tens objects and with hundreds and tens objects.

7. When read numbers like 130, to write the number and say it is 1 hundred and 3 tens.

8. When read numbers like 130, to write the number as 13 tens.

9. When given any of the three symbolic forms:

130

___ hundreds, ___ tens

___ tens , to

a. tell the other forms orally,
b. complete the other symbolic forms.

Lesson 22 was titled "3-digit numerals." The broad objective of this lesson was to provide similar activities as those in lesson 21, but to extend these activities to larger numbers. Lesson 22 extended the number activities mostly to numbers from 100 to 200. The stated objectives for the enactive level of instruction were as follows:

1. Given numbers like 123, 169, etc., to
   a. show the number with tens objects and ones objects and with hundreds, tens, and ones objects,
   b. tell that 123, for example, is equal to
      i) 1 hundred, 2 tens, and 3 ones;
      ii) 12 tens and 3 ones; and
      iii) 123 ones.

2. Given numbers 100, 200, ..., 900 displayed with hundreds objects, to
   a. count, for example, to 300 by hundreds, by tens;
   b. tell, for example, that 300 equals 30 tens;
   c. tell that 30 tens, for example, is equal to 300 ones.

3. Do rational counting from 190 to 215 by ones.

4. Given numbers which are multiples of 100, to count from one multiple of 100 to the successive multiple of 100 by tens.

5. Given numbers like 210, 250, 260, to
   a. display the numbers with tens objects,
   b. count to the number by tens,
   c. count to the number by hundreds and tens,
   d. trade 10 tens for one hundred.

6. Given numbers like 220, 250, 270, to
   a. show the number with hundreds objects and tens objects,
   b. show the number with objects which are 1, 2, 10, and 20 more (and less).

7. Given numbers like 215, 269, ..., to
   a. show the number with hundreds, tens, and ones objects;
   b. show the numbers which are 1, 2, 10, and 20 more (and less).

Appropriate modifications of the statements to reflect iconic and symbolic activity define the objectives for the iconic and symbolic phases of instruction.

LOG EXCERPTS

Lesson 21

All seemed excited about the new group of 100 sticks (bundle of bundles).

Carrie and Cubby had trouble with "more" and "less" concepts and did not always know whether to add sticks or take away sticks. Carrie would sometimes add a bundle to show one more instead of a stick. All seemed to understand that when 10 bundles are traded for a bundle of bundles, the number remains the same.

All children were able to count 13 bundles and say that there were 130 sticks in all.

All could show 170 with bundles, trade for 1 bundle of bundles and have 7 bundles left over.

Joe Benny was the only one who knew immediately that there were 17 tens in all. Others had to point and count to obtain answer.

When Cubby saw the picture of 10 sets of 10 sticks he said, "That looks like 100 sticks." Then Phil said, "That's the same as a bundle of bundles."

Children (except Carrie) did well with concepts of 10 more, 10 less, 1 more, 1 less, 2 more, 2 less.

To order pictures from 110-200, Opie was the first one to understand what to do. He appeared to zero in on the fact that he needed to count only the bundles after the first group of ten bundles. Phil, Joe Benny, and Cubby seemed to count the total sticks shown on each picture.

Carrie is not sure about increasing a number by a specific number. She knows that she must add some sticks to increase a number but does not always know what to add.

Children were given written exercises like:

170 = ___ hundred, ___ tens, and

190 = ___ tens.

Some errors noted were:

170 = 1 hundred, 17 tens

200 = 1 hundred, 2 tens

200 = 2 tens

200 = 1 hundred, 1 ten

Washington and David answer all questions easily; cue in on 20 more/less as 2 tens more/less.

Duke seemed to parrot the phrase "10 tens" without grasp of meaning.

After adding/subtracting blocks on more/less, Duke always counted his blocks from zero to new total, even if he added 1 unit, he recounted the tens. In reply to "count the tens" and "count the longs," Duke counted 10, 20, 30, ...,.

David and Washington can go from counting "13 tens" to saying "130" immediately.

Susan, David, and Washington can go from 170 to 17 tens and vice versa without recounting.

Given choice, most prefer to use flat and longs to show number (e.g., 170) rather than all longs.

Susan and David can make "inventive" trades; e.g., 1 flat 2 longs traded to 1 flat, 20 units; 1 flat, 5 longs traded to 1 flat, 4 longs, 10 units.

Duke and Celia confuse 2 more/less with 20 more/less.

For picture of 100 units, Duke said there were 10 in each group (without counting), and after we counted 1, 2, ..., 10 groups said that was 100 altogether. Good guesses or maybe "10 tens" has some meaning for him finally.

More/less tasks are faster with picture of original number for children to refer to. Interesting responses: 20 more than 87--Washington thought a second and said "107"; Susan, David, and Celia showed 2 more longs but only Celia needed to count to know 107; Susan traded for flat. 2 less than 71--Susan traded first, then others; Duke took away 2 longs.

In writing numerals from 100 to 125, for 110 Susan wrote 101; Celia 1010.

All but Duke can go from statement like 12 tens and tell how many in all. All but Duke and Celia can go from total to ____ tens.

I used an open-end abacus to show 100; kids were excited about the height. Paul began counting ones by tens and everyone joined in naturally.

More and less were difficult. When I asked them to show the number that is 10 more, everyone stared and finally Chris said, "10 more what?" Omar eventually put on 10 ones, counted beads and got answer; others followed suit. We went to multiples of ten only and practiced. Annie was completely lost, didn't know whether to put on or take off beads. Kate was about the same.

Everyone had trouble counting by tens after 100, said, "100, 101, 102."

Question about "How many tens in 120?" for example, was very difficult.

No one could write the numeral for one hundred nine, yet they could count beads on abacus and say one hundred nine.

Kids had lots of trouble writing numerals. Chris wrote (correctly) 101 for abacus display, others wrote 1001, 1002. I tried to relate digits to abacus rods, which worked fine until we showed 1 hundred, 10 ones, and everyone wrote 1010.

On writing numbers by tens, 100-200, Chris was the only one who did it right; Annie: 100, 110, 120, 130, 114, 115, ...; Omar: 100, 110, 112, 113, ...; Chris: 100, 101, 102, ...; Kate wrote 100-190 correct, then 102 for 200; Paul: 100, 1010, 11102, etc., then wrote 1200 for 200.

On several worksheets which showed a picture of an abacus display, children had a great deal of trouble writing the corresponding 3-digit numeral, especially for exercises with 17 tens on the abacus and children were to write the number 170.

Kids excited about getting new materials (bundles-of-bundles, flats). Alex is very hesitant as to how many ones in one hundred.

Kids getting used to 10 tens being 100 but that 18 tens is 180 is still difficult.

Karl will guess that 12 tens are 120 but wants to check it with a display. Calvin has no trouble with 12 tens or 13 tens but hesitates about 18 tens.

All seem to be able to give manipulative displays for multiples of 10; 0-200. With the display present all can tell how many tens. Without the display April and Calvin, but not Karl and Alex, can do it if they stop and think--not automatic.

All agree that different-looking displays (e.g., 1 hundred, 5 tens; or 15 tens; or 1 hundred, 4 tens, 10 ones) using the same manipulative can show the same number.

An exercise requiring children to order picture to "show counting by tens" was very instructive, seemed to crystallize things for Karl, Calvin, and April.

Calvin and April quickly caught on to writing 3-digit numbers. Karl and Alex translate a number into how many hundreds, tens and ones; then write.

Karl wrote 101 for 110; Alex wrote 10010 for 110.

To complete _15_ tens

____ hundreds, ____ tens,

____ , Karl consistently wanted to write

_15_ tens

_100_ hundreds, _5_ tens

_150_.

"Ten more" went fairly well, but "ten less" was like pulling teeth.

Mary and Tommy had trouble realizing that we would add blocks to make more and take away blocks to make less. They also had trouble with whether to add one long or one unit for 1 more/10 more, and similarly for less.

I had 13 longs to display 130; we counted longs by ones to find out how many tens in 130. When asked how many longs or tens, they said 130 or 113--very hard for them to see this relationship.

Exercise 4, where they ordered pictures to show counting by tens from 100 to 200 was significant--helped them see the pattern.

After 119, tendency to write 1120--using a manipulative helped.

On a worksheet--a picture is given, children are asked to use the picture and complete the following:

____

____ hundreds, ____ tens

____ tens.

The second line caused lots of problems. Majority of errors look like this:

_130_

_1_ hundred, _13_ tens

_30_ tens

_130_

_1_ hundred, _30_ tens

_30_ tens

_130_

_1_ hundred, _13_ tens

_30_ tens

Then came Worksheet 14 (with symbolic sentences like 17 tens = ____ hundred, ____ tens; 19 tens = ____ tens, ____ hundreds; no pictures present --disaster.

Abel convinced me he understands the three relationships but just can't handle all the symbols.

Lesson 22

From a stick display Joe Benny and Phil could tell how many (hundreds, tens, and ones), (hundreds, ones), (tens, ones) were in the number represented.

Counting bundles of bundles by tens (i.e., as 10 tens, 20 tens, ...) seemed difficult at first but eventually children caught on. Children wanted to say 10, 20, 30, etc. without using the word tens.

Seemingly, children have to be prompted to trade either 10 ones for 1 bundle, or 10 bundles for 1 bundle of bundles. Trading does not occur as automatically as it once did.

Children confuse the questions "How many tens?" and "How many in all?"

When asked to show 219, Phil first showed 2 bundles of bundles, 9 bundles, then corrected himself.

Carrie could not determine the number represented by the picture of 19 bundles, 3 sticks arranged as one set of 10 bundles, 9 bundles, and 3 sticks.

All exercises which called for me showing a picture and having the children find the picture to show 100 more, 10 more, 1 less, 10 less, 1 more, 100 less were extremely difficult. Strangely enough, the comparable exercises at the enactive level went very well. Apparently children were not ready for these kinds of activities at the iconic level.

Counting flats by tens (10 tens, 20 tens, ...) less meaningful for children (especially Duke) than counting either by ones or hundreds.

On counting tasks, all have trouble at places like 199-200, 290-300. Susan, David, and Washington can say next number without seeing trade, but others must see the trade first.

Today Duke made errors in 1/10 more/less but after he got the display correct, he didn't count from 0, he counted up or down from the original number!

Counting on by ones, tens, or hundreds from number (e.g., 243) in picture display--all children had most trouble counting by tens. David, Washington, and Susan could but correcting errors. Duke and Celia needed block displays and trades in order to bridge hundreds in counting by tens.

More and less went well, except that Annie is still uncertain whether to add or take away a bead.

Counting 190-215 by ones: kids counted 1 hundred and 9 tens, had trouble counting 191, 192, then okay 'til 199.

No one knew what to call 1 hundred, 9 tens, 10 ones. Finally Chris saw that 9 tens = 90, 10 ones = 10, 90 + 10 = 100, so it was 200. No one could go on to 201, then okay until 209. Chris said 210 was next. Omar argued hotly that it was 300. We counted again but Omar couldn't say 209-210." He kept saying, "I don't see it, I don't see it!" Finally I put my hand between the two sets of 10 ones so that 200 was below and 10 above my hand. Omar counted and agreed that 200 was below, then he counted 10 above. I asked him what 200 and 10 more was and he said, "Okay, now I see it."

More/less: Omar had 506, I asked him for 10 less and he hesitated. Paul and Claire both picked up their abaci, showed 506, traded correctly to show me what Omar should do.

When I showed 90 tens, Chris remarked if we had "ten hundreds" it would be "one hundred tens." He and Omar liked the play on words, reversing them. I told them it would be 1000. They got excited.

This lesson has really brought out great interest in using abacus. Not since trading was first introduced have kids enjoyed using abacus so much.

The place-value-chart was very effective in helping kids learn to circle three-digit numerals.

After getting 12 bundles and 3 ones, rather that separating as 100, 20, 3, I broke all bundles apart (123 ones) and told kids to arrange these so I could tell how many easily. Good exercise. Alex really got into it. Finally had 10 bundles, 2 bundles, 3 sticks.

On more/less items, April and Calvin say how many it'll be, then adjust their display. Alex adjusts his display, then counts to see how many. Calvin and April good at interchanging n tens or (n x 10) ones. Karl okay on 40 tens (said 400) but kept saying 90 tens was 109 or 119--may be a verbal problem.

None of the children knew what a number between 500 and 600 would sound like. Karl said "287 + 369," Calvin said, "730."

Alex easily deals with Dienes blocks but has trouble seeing a bundle of bundles as 10 bundles.

Finding 10 less than 309 from a picture was hard. When blocks were brought to illustrate, Calvin knew what to do.

All doing very well at writing. Even Alex likes writing. Kept saying he couldn't be tricked by mixed up order (e.g., 4 tens, 5 ones, 3 hundreds = ____).

Writing numerals to show counting by tens starting at, for example, 407, going slowly but smoothly. Even Alex getting this. When a display is used at first to see the pattern, Karl and Alex okay. Calvin and April can do it without the display.

I read 1 hundred, 9 ones, 3 tens and asked the kids to show with colored chips (hundred = blue, one = white, ten = red). Abel and Tommy put objects out in order they were called; e.g., 1 blue, 9 white, and 3 red; but they read it correctly.

A task was to show 22 tens with blocks--no one could do it. When I held up a long and asked, "How much?" they said, "1 ten"--I said, "Show me 22 tens." Only Betty did it. Finally everyone did it and we counted to get 220. Then, when I asked "How many tens in 220?" no one could remember. Betty figured it out--the others had to count.

They all had difficulty starting at a number and counting by hundreds, tens, and ones. Putting objects out helped, but even then, only Betty and Tommy seemed sure of themselves and could eventually do it without objects.

Bridging the hundreds is difficult for them all.

Worksheet with exercises like: ____tens

____ones 635

____hundreds

went better than expected--they actually read the words. Mary noticed that on ____tens

____ones 333

____hundreds

she didn't have to read the words because they are all threes.

V. REPORT ON EVALUATIONS

EVALUATION INTERVIEW 1--2-DIGIT NUMERATION

The first evaluation interview was concerned with an evaluation of the children's concept of number for numbers named by 2-digit numerals. In the interview children were presented with situations which gave them an opportunity to show their knowledge of place value. Questions were presented in symbolic, oral, and manipulative modes and children were asked to respond to questions in these modes as well. The children were asked to read 2-digit numerals, to explain the meaning of each of the digits in a 2-digit numeral, to use manipulatives to show a 2-digit number and to give numbers which were 10, 1, and 1-ten more than given numbers.

The interview questions were prerecorded for video presentation to the children. During the interview the interviewer would start and stop the videotape and proceed with further questioning when appropriate. The detailed outline of the interview is given in Appendix B. A video recording was made of each individual interview. Following the interview, the recordings were viewed and each child's performance scored by the evaluators according to a scoring procedure which they designed. The scoring procedure assigned to each child two scores, one was a score which reflected the child's ability to explain why he gave the response. The latter was called an understanding score.

Shown in Table 2 is the profile sheet used to record the information about a child's performance and also the number and percent of the children in each group who were judged to have performed satisfactorily on that item.

Table 2

Student Profile Sheet for Interview 1 with the number and percent of students who responded correctly to each item by groups

	Item	Group					
		*U1 (5)	U2 (5)	U3 (6)	M (4)	2M (4)	C (5)
(1)	Read 35 as	3 (60)	3 (60)	5 (83)	3 (75)	3 (75)	5 (100)
	Read 53 as	3 (60)	3 (60)	5 (83)	3 (75)	3 (75)	5 (100)
	Why 53 (___)	3 (60)	1 (20)	4 (67)	2 (50)	4 (100)	4 (80)
	5 means	4 (80)	3 (60)	4 (67)	4 (100)	4 (100)	3 (60)
	3 means	5 (100)	3 (50)	5 (83)	4 (100)	4 (100)	2 (40)
(2)	Thinks of 24 as	3 (60)	1 (20)	2 (33)	2 (50)	3 (75)	1 (20)
	After 20 + 4, thinks						

(continued next page)

Table 2 (continued)

(3)	___ tens and ___ ones	4 (60)	3 (60)	6 (100)	4 (100)	4 (100)	2 (40)
		5 (100)	3 (60)	6 (100)	4 (100)	4 (100)	
	___(b) and ___(c)	5 (100)	3 (60)	6 (100)	4 (100)	4 (100)	
	___(a) Why (a)___	2 (40)	1 (20)	2 (33)	1 (50)	0 (0)	1 (20)
	Why (b)___	2 (40)	1 (20)	4 (67)	2 (50)	1 (25)	1 (20)
	Why (c)___	1 (20)	2 (30)	5 (83)	1 (25)	1 (25)	2 (40)
(4)	ten more than 20: wrote___ Why___	0 (0)	1 (20)	1 (67)	2 (50)	0 (0)	3 (60)
		1 (20)	1 (20)	1 (67)	2 (50)	0 (0)	3 (60)
	one more than 32: wrote___ Why___	1 (20)	1 (20)	2 (33)	2 (50)	0 (0)	3 (60)
		1 (20)	1 (20)	2 (33)	2 (50)	0 (0)	3 (60)
	one ten more than 30: wrote___ Why___	1 (20)	1 (20)	1 (67)	2 (50)	0 (0)	3 (60)
		1 (20)	1 (20)	1 (67)	2 (50)	0 (0)	4 (80)
(5)	How many pieces of candy? wrote_____	3 (60)	2 (40)	2 (33)	3 (75)	4 (100)	5 (100)
	said_____	3 (60)	2 (40)	3 (33)	2 (50)	4 (100)	5 (100)
	How knows	3 (60)	2 (40)	3 (33)	2 (50)	2 (50)	5 (100)
(6)	Configuration	4 (80)	3 (60)	4 (67)	4 (100)	4 (100)	4 (80)
	___________	4 (80)	3 (60)	2 (33)	4 (100)	4 (100)	5 (100)
	Why 23 ___________						
(7)	Configuration	5 (100)	2 (40)	3 (50)	4 (100)	4 (100)	5 (100)
	___________	5 (100)	2 (40)	2 (33)	4 (100)	4 (100)	5 (100)
	Number read ___________	4 (80)	7 (20)	2 (33)	3 (75)	4 (100)	4 (80)
	Why (________)						
(8)	Showed 27 - Yes No _____	3 (60)	2 (40)	4 (67)	3 (75)	3 (75)	5 (100)
	Showed 28 - Yes No _____	3 (60)	2 (40)	3 (50)	4 (100)	2 (50)	5 (100)
	Read 28 - Yes No _____	4 (80)	3 (60)	3 (50)	3 (75)	3 (75)	4 (80)
	Showed 38 - Yes No _____	2 (40)	3 (60)	3 (50)	3 (75)	3 (75)	4 (80)
	Read 38 - Yes No _____	2 (40)	3 (60)	2 (33)	4 (100)	4 (100)	4 (80)

*The number in parentheses indicated the number in the group.

In Table 3 are presented adjusted and unadjusted means for the five experimental groups. The means are adjusted to reflect differences among the groups on the KeyMath Diagnostic Test and the Otis-Lennon Mental Ability Test, Elementary 1 Level, Form J. In an intuitive sense the adjusted means equalize the groups for KeyMath scores and IQ scores. The adjusted means for the understanding scores order the groups from highest to lowest performance in the order 2M, U1, U2, U3, and M. For the skill socres, the adjusted means order the groups as 2M, U2, M, U1, and U3. Group 2M was highest on both skill and understanding while Group U3 (the abacus group) was among the lower two for both skill and understanding scores.

Also presented in Table 3 is the F and the significance level of the F from the analysis of covariance which was done to determine whether there are statistically significant differences between any of the adjusted group means. The analysis was conducted using KeyMath test scores and IQ test scores as covariates. The information suggests that the data trend for skill scores has associated with it a probability of .912, that the observed differences are due to treatment effects and are not due to a chance occurrence. The data trend observed for the understanding scores has a very low probability (.60) that the observed differences are due to treatment (instructional differences) effects rather than to a chance occurrence. This investigator urges that extreme caution must be exercised in interpreting these statistics. The number of subjects in each treatment group was so small as to raise the question about whether the statistics given in Table 3 are meaningful. There is no agreement among statisticians about the effect of small samples of the F statistic. Computed R^2's of .67, .70, and .73 suggest that the combination of KeyMath Test scores and IQ scores account for 67, 70, and 73 percent of the variation in the skill, understanding, and total scores.

Table 3

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for Interview 1 skill and understanding scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ scores

Group	Skill $\bar{X}$	Skill $A\bar{X}$	Skill F	Skill P	Understanding $\bar{X}$	Understanding $A\bar{X}$	Understanding F	Understanding P
U1	12.20	13.37			7.20	8.48		
U2	15.20	15.35			7.80	7.85		
U3	11.33	11.34			6.33	6.22		
M	17.75	14.89			8.75	6.06		
2M	16.50	17.90			7.50	8.72		
			2.42	.088			1.08	.40

Another word of caution is in order at this point about drawing a conclusion that the 2M instructional method is superior and that the U3 instructional method is inferior. The possibility exists that the 2M group contains a child whose expected performance based on pre-test scores was less than his actual performance. This could, in a small group, substantially affect the mean of the entire group. Similarly if a child in the U3 group had scored high on the pre-tests, but performed below expectation on the interview, then this could substantially depress the mean for that group.

In Table 4 are presented adjusted and unadjusted means for five experimental groups and the control group. The means are adjusted to reflect difference among the groups on the KeyMath Diagnostic Test. The adjusted means for the skill and understanding scores order the groups from highest to lowest performance in the order 2M, U2, M, U1, U3, C and 2M, U2, U1, U3, M, C. For both the skill and understanding scores each of the experimental groups performed as well or better than the control group.

Table 4

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for Interview 1 skill and understanding scores by groups and the significance level of the F-statistic for the 1 x 6 analysis of covariance with KeyMath and IQ scores

	Skill				Understanding			
GROUP	$\bar{X}$	$A\bar{X}$	F	P	$\bar{X}$	$A\bar{X}$	F	P
U1	12.20	13.72			7.20	8.77		
U2	15.20	16.49			7.80	9.13		
U3	11.33	12.54			6.33	7.58		
M	17.75	15.83			8.75	6.78		
2M	16.50	18.29			7.50	9.34		
C	16.80	12.65			8.40	4.13		
			2.40	.07			1.73	.17

Some concluding remarks which reflect the investigator's interpretation of these data are:

1. The abacus taken by itself does not serve as an adequate manipulative aid for developing number concepts in children for numbers named by 2-digit numerals.

2. The superior performance of the 2M instructional group suggests the possibility that a multi-embodiment approach is superior to a uni-embodiment approach. However, this remark must be greatly tempered by the fact that the M group's performance, as indicated by the adjusted means, was lowest for the understanding scores and in the middle for skill scores.

We turn our attention next to another type of data which reflects performance of the children in the several instructional groups. An analysis of the children's responses to the questions presented in Interview 1 suggest that children's responses to questions about numbers can be categorized as reflecting one of the following interpretations of a number named by 2-digit (or 3-digit) numerals:

1. As tens and ones, for example 24 as 2 tens and 4 ones. Similarly for 3-digit numerals: as hundreds, tens and ones.

2. As a multiple of ten and ones, for example 24 as 20 and 4 ones. Similarly for 3-digit numerals: as multiples of one hundred and ten and ones.

3. As ones, for example 24 as 24 ones.

4. As separate entities, for example, 24 as a 2 and a 4.

5. As tens and ones, with ones greater than nine, similarly for 3-digit numerals: as hundreds, tens and ones with the number of tens or ones greater than nine.

6. Other, that is, responses that are not classifiable into any one of the other five categories.

The fact that the children's responses were observed to fall into these categories was not a surprise since it was an instructional objective to teach children to view "2-digit numbers" in each of the first three categories.

A coding scheme was developed by the investigator to code children's responses into the identified categories. After the responses were coded, including an indication of whether the response was considered correct within the category, the number of attempts and number of correct responses for each subject in each category were recorded. Within category 4 a response was labeled as "correct" provided that the child kept the two digits separated in his thinking as opposed to "joining them" and thinking of 24, for example, as "2 and 4 and that's 6." Then a number concept index indicating the number of correct responses as a percent of the total number of attempts was computed for each child and for each group. A summary of this information is given in Table 5 (on the following page.)

It is apparent that this number concept index for Group M - 44.7, 10.6, 12.8, 4.3 - represents a superior number concept than the index for group U3 - 18.3, 16.9, 9.1, 2.6, for example. Although the investigator has no definitive way to determine which of two such indices represent a statistically significant superior performance, it does appear that the instructional treatment U3 was definitely inferior for developing an adequate number concept for children. The higher ability children in the experimental group (Exp. H^+ U H^-, Table 5) and in the control groups

are comparable on pre-test scores, but the number concept index for the Exp $H^+ \cup H^-$ appears to be considerably higher than that for the control group.

Table 5

Summary of coded responses indicating the number of correct responses in categories 1, 2, 3, and 4 as a percent of the total number of responses coded.

	Category			
Group	1	2	3	4
U1	30.4	7.3	10.1	2.9
U2	40.8	16.3	4.1	6.1
U3	18.3	16.9	9.1	2.6
M	44.7	10.6	12.8	4.3
2M	39.6	3.8	1.9	0
C	27.1	11.9	18.6	10.1
*Exp $H^+ \cup H^-$	50.5	19.4	9.7	1.01

*The group of students in the experimental groups classified as high according to their scores on the pre-tests.

A particular observation made by the investigator and group instructors during the day-to-day conduct of the experiment concerned children's ability to give correct responses using a manipulative as compared with their ability to give symbolic responses to similar questions. During information discussions, it was frequently stated by the group teachers and the investigator that when the response mode was manipulative the lower ability children performed as well as the high ability children.

In Table 6 (found on the following page) the total number of correct responses for the high, middle, and low groups of experimental group children is given as a percent of the total number of responses attempted in these categories.

While these data suggested that the observation made above is not a true statement, there is indication that the lower ability children do perform comparatively better at the manipulative level than at the symbolic level. The differences between the percent correct for the Exp $H^+ \cup H^-$ and the Exp $L^+ \cup L^-$ groups for the manipulative and written response categories was 34% and 60%, respectively. That is, when the response made was symbolic, the "high" students gave 60% more correct responses than the "low" students; when the response made was manipulative, this difference was reduced by nearly 1/2 -- to 34% more correct responses for "high" than "low" students.

The implication of these data for teacher who wish to give their children a greater feeling of success is clear.

Table 6

Correct oral, manipulative, and written responses as a percent of total responses attempted.

Group	Response Category		
	oral*	manipulative	written
Exp H^+ U H^-		93%	83%
Exp M^+ U M^-		49%	22%
Exp L^+ U L^-		59%	20%

*The total number of oral responses requested of subjects in the interview was so small as to make any comparisons meaningless.

EVALUATION INTERVIEW 2--ADDITION AND SUBTRACTION FOR NUMBERS NAMED BY 2-DIGIT NUMERALS, NO REGROUPING.

The second evaluation interview was concerned with an evaluation of the children's concepts of addition, subtraction, and order for numbers named by 2-digit numerals. In the interview children were presented problems dealing with addition, subtraction, and order. Addition and subtraction problems were presented symbolically, iconically, and orally. The order problems were presented symbolically. For addition and subtraction problems presented symbolically or orally, the children were asked to solve the problem and explain their procedure in arriving at the answer. For addition and subtraction problems presented iconically, children were directed to write the problem for the picture, find the answer to the problem a.. explain how the problem and picture "go together." One addition problem and one subtraction problem involved regrouping. These items were included to investigate children's ability to transfer knowledge to this type of problem. The order questions required the children to indicate which of two numbers was more or less. And also to complete a sentence by inserting < or >, to make a true sentence, and to explain why one of the two numbers is more or less.

The evaluation questions were determined in advance by the evaluators, according to the objectives of the written teaching materials. The questions and test materials were presented to the children according to prewritten script. Packets of problems were organized and presented to the children; for example, the first six items of the evaluation were concerned with addition. The children were presented with this packet of six problems and the directions and questions for each item were

essentially the same, except that the last two items in the packet required the children to use manipulative objects to aid in their explanation of how the problem was worked.

The complete script for the structured interview evaluation is contained in Appendix C.

Shown in Table 7 (the next 4 pages) is the profile sheet used to record the information about each child's performance and also the number and percent of the children in each group who were judged, by the evaluators, to have performed satisfactorily on each item.

In Table 8 are presented adjusted and unadjusted means for the five experimental groups for skill and understanding scores derived from Interview 2. The means were adjusted as described for Interview 1 scores.

Table 8

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for Interview 2 skill and understanding scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ scores.

	Skill				Understanding			
Group	$\bar{X}$	$A\bar{X}$	F	P	$\bar{X}$	$A\bar{X}$	F	P
U1	31.20	33.94			21.00	24.04		
U2	31.20	30.82			25.80	25.99		
U3	28.83	28.09			22.50	22.30		
M	26.75	22.64			24.00	17.30		
2M	26.25	28.42			18.50	21.47		
			1.56	.229			.61	.999

Table 7

Student Profile Sheet for Interview 2 with the number and percent of students who responded correctly to each item by groups.

	ITEM		U1(5)	U2(5)	U3(6)	M(4)	2M(4)	C(5)
(1)	Read problem	0 1	4(80)	2(40)	5(100)	2(50)	3(75)	5(100)
	5 + 23	0 1	5(100)	5(100)	5(100)	4(100)	3(75)	5(100)
	Explanation	0 1	5(100)	5(100)	3(50)	2(50)	3(75)	4(80)
(2)	Read problem	0 1	5(100)	5(100)	5(67)	3(75)	3(75)	4(80)
	30 + 40	0 1	4(80)	5(100)	6(100)	3(75)	4(100)	5(100)
	Explanation	0 1	4(80)	4(80)	4(67)	2(50)	3(75)	4(80)
(3)	Read problem	0 1	5(100)	5(100)	5(93)	2(50)	3(75)	4(80)
	40 + 23	0 1	5(100)	5(100)	6(100)	3(75)	3(75)	5(100)
	Explanation	0 1	5(100)	4(80)	4(67)	2(50)	3(75)	4(75)
(4)	Read problem	0 1	5(100)	4(80)	6(100)	3(75)	3(75)	3(60)
	42 + 7	0 1	5(100)	4(80)	5(93)	4(100)	3(75)	4(80)
	Explanation	0 1	3(60)	3(60)	3(50)	3(75)	3(75)	3(60)
(5)	Read problem	0 1	5(100)	5(100)	5(93)	2(50)	3(75)	4(80)
	46 + 23	0 1	5(100)	5(100)	6(100)	4(100)	3(75)	5(100)
	Explanation	0 1	4(80)	4(80)	4(67)	3(75)	4(100)	4(80)
	Show with aid	0 1 2	0 (0) | 5 (100)	0 (0) | 5 (100)	0 (0) | 5 (100)	0 (0) | 4 (100)	0 (0) | 4 (100)	0 (0) | 4 (80)

Table 7 (part 2)

	ITEM		U1(5)	U2(5)	U3(6)	M(4)	2M(4)	C(5)
	Read problem	0 1	5(100)	5(100)	4(67)	3(75)	3(75)	4(80)
(6)	27 + 35	0 1	3(60)	3(60)	2(33)	1(25)	0(10)	4(80)
	Explanation	0 1	3(60)	3(60)	3(50)	1(25)	2(50)	3(60)
	Aware of regrouping	0 1	2(40)	3(60)	3(50)	2(50)	1(25)	3(60)
	Show with aid	0 1 2	2 (40) | 2 (40)	2 (40) | 2 (40)	5 (93) | 0 (0)	1 (25) | 1 (25)	3 (75) | 0 (0)	2 (40) | 0 (0)
	Read problem	0 1	5(100)	5(100)	6(100)	2(50)	3(75)	4(40)
(7)	37 − 4	0 1	3(60)	5(100)	6(100)	3(75)	4(100)	5(100)
	Explanation	0 1	3(60)	5(100)	3(100)	2(50)	3(75)	4(80)
	Read problem	0 1	5(100)	5(100)	5(93)	2(50)	3(75)	4(80)
(8)	56 − 6	0 1	4(80)	5(100)	6(100)	4(75)	4(100)	5(100)
	Explanation	0 1	3(60)	5(100)	3(50)	2(50)	3(75)	4(80)
	Read problem	0 1	5(100)	5(100)	4(67)	3(75)	3(75)	4(80)
(9)	69 − 53	0 1	4(80)	5(100)	6(100)	4(100)	4(100)	5(100)
	Explanation	0 1	4(80)	4(80)	5(93)	3(75)	4(100)	4(80)
	Show with aid	0 1 2	0 (0) | 3 (60)	1 (20) | 4 (80)	0 (0) | 5 (93)	0 (0) | 3 (75)	0 (0) | 4 (100)	2 (40) | 1 (20)
	Read problem	0 1	5(100)	5(100)	4(67)	2(50)	3(75)	4(80)
(10)	53 − 24	0 1	0(0)	0(0)	0(0)	0(0)	0(10)	3(60)
	Explanation	0 1	0(0)	2(40)	3(50)	0(0)	1(25)	2(40)
	Aware of regrouping	0 1	0(0)	1(20)	3(50)	3(75)	1(25)	2(40)
	Show with aid	0 1 2	0 (0) | 0 (0)	0 (0) | 1 (20)	1 (17) | 0 (0)	0 (0) | 0 (0)	0 (0) | 0 (0)	0 (0) | 1 (20)

Table 7 (part 3)

(11)	24 + 32	0 1	2(40)	3(60)	2(33)	2(50)	2(50)	3(60)
	Explanation	0 1	1(20)	4(80)	2(33)	2(50)	0(10)	0(0)
(12)	23 + 5	0 1	3(60)	2(40)	4(67)	3(75)	2(50)	4(80)
	Explanation	0 1	1(20)	3(60)	3(50)	4(100)	1(33)	3(60)
(13)	32 + 6	0 1	3(60)	2(40)	2(33)	3(75)	1(33)	4(80)
	Explanation	0 1	1(20)	3(60)	3(50)	2(50)	2(50)	4(80)
(14)	45 - 32	0 1	1(20)	2(40)	2(33)	1(33)	1(33)	1(20)
	Explanation	0 1	0(10)	2(40)	2(33)	1(33)	0(0)	0(0)
(15)	27 - 4	0 1	1(20)	3(60)	3(50)	3(75)	2(50)	4(80)
	Explanation	1 1	1(20)	3(60)	3(50)	3(75)	2(50)	3([illegible])
(16)	Wrote problem	0 1	5(100)	5(100)	4(67)	4(100)	4(100)	2(40)
	Explanation	0 1	4(80)	4(80)	3(50)	3(75)	4(100)	2(40)
(17)	Wrote problem	0 1	5(100)	5(100)	4(67)	3(75)	4(100)	2(40)
	Explanation	0 1	4(80)	4(80)	4(67)	3(75)	3(75)	2(40)
(18)	Read problem	0 1	3(60)	5(100)	[illegible](67)	2(50)	4(100)	4(80)
	43 + 15	0 1	5(100)	3(60)	3(50)	3(75)	4(100)	4(80)
	Show with aid	0 1 2	2 3 4[illegible] [illegible]	1 2 20 40	0 3 0 50	1 2 2 (33) (5[illegible]) (50)	2 1 (50) (20)	2 [illegible]

Table 7 (part 4)

(19)	Read problem	0 1	5(100)	5(100)	5(93)	2(50)	3(75)	4(80)
	45 -12	0 1	3(60)	4(80)	4(67)	2(50)	4(100)	5(100)
	Show with aid	0 1 2	1 2 (20) (40)	2 2 (40) (40)	0 3 (0) (50)	1 2 (33) (50)	2 2 (50) (50)	0 4 (0) (80)
(20)	Read problem	0 1	4(80)	5(100)	4(67)	3(75)	3(75)	4(80)
	Relates to picture	0 1	3(60)	4(80)	6(100)	3(75)	2(50)	3(60)
	How/changes	0 1	3(60)	4(80)	3(50)	2(50)	1(25)	2(40)
(21)	Read problem	0 1	4(80)	5(100)	4(67)	3(75)	3(75)	4(80)
	Relates to picture	0 1	1(20)	3(60)	3(50)	3(75)	1(75)	1(20)
	How/changes	0 1	1(20)	3(60)	2(33)	3(75)	0(0)	0(0)
(22)	Why harder	0 1	0(10)	1(20)	3(50)	3(75)	2(50)	3(60)
(23)	Why harder	0 1	0(0)	1(20)	1(17)	1(25)	2(25)	1(20)
(24)	Read numbers	0 1	4(80)	5(100)	6(100)	3(75)	3(75)	4(80)
	Circled more	0 1	4(80)	5(100)	6(100)	4(100)	3(75)	5(100)
	Explanation	0 1	1(20)	4(80)	6(100)	4(100)	1(25)	5(100)
	Wrote	0 1	1(20)	1(20)	5(93)	3(75)	0(0)	3(60)
(25)	Read numbers	0 1	3(60)	5(100)	6(100)	3(75)	3(75)	5(100)
	Circled less	0 1	3(60)	2(40)	5(93)	3(75)	3(75)	5(100)
	Explanation	0 1	1(20)	3(60)	5(93)	3(75)	1(25)	5(100)
	Wrote	0 1	3(60)	1(20)	3(50)	2(50)	3(75)	2(40)
	Show with aid	0 1	2(40)	3(60)	3(50)	3(75)	1(25)	2(40)

The adjusted means for the skill scores order the groups from highest to lowest performance in the order U1, U2, 2M, U3, M. For the understanding scores the adjusted means order the groups as U2, U1, U3, 2M, M. Also presented in Table 8, 2 is the F-statistic and the significance level of the F from the analysis of covariance which was done to determine whether there are statistically significant differences between any of the adjusted group means. The analysis was conducted using KeyMath and IQ test scores as covariates. This information suggests that the data trend apparent for skill scores has associated with it a probability of .771 that the observed differences are due to instruction and are not a chance occurrence. The data trend for the understanding scores has a very low probability (at most .001) that the observed differences are due to instruction rather than being a chance occurrence.

Similar information concerning the analysis of data including the control group is given in Table 9 below.

Table 9

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for Interview 2 skill and understanding scores by groups and the significance level of the F-statistic for the 1 x 6 analysis of covariance with KeyMath scores.

Group	Skill				Understanding			
	$\bar{X}$	$A\bar{X}$	F	P	$\bar{X}$	$A\bar{X}$	F	P
U1	31.20	33.62			21.00	23.63		
U2	31.20	33.25			25.80	28.03		
U3	28.83	30.76			22.50	24.60		
M	26.75	23.68			24.00	20.65		
2M	26.25	29.10			18.50	21.60		
C	31.60	24.97			21.80	14.59		
			1.20	.340			.69	.999

In addition to the interview evaluation conducted by the evaluators, the investigator, in conjunction with the group teachers, prepared paper and pencil tests to measure children's computational skill. Separate tests were written for addition and subtraction. The addition-without-regrouping test was in two parts. The first part had 20 problems

given under power conditions, the second part had 10 problems given under speed conditions. This test was scored on the basis of 30 points. In Table 10 (given below) are presented means and adjusted means for the addition-without-regrouping total test scores.

Table 10

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for addition without regrouping total scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ scores and the 1 x 6 analysis with KeyMath scores.

Group	$\bar{X}$	1x5 $A\bar{X}$	1x5 F	1x5 P	1x6 $A\bar{X}$	1x6 F	1x6 P
U1	27.40	27.88			28.37		
U2	25.20	25.51			26.02		
U3	25.50	25.75			26.27		
M	28.50	26.47			27.28		
2M	26.50	27.17			27.64		
C	26.20				26.57		
			.304	.999		.36	.999

Adjusted means for the 1 x 5 and 1 x 6 analyses order the groups from highest to lowest performance in the order U1, 2M, M, U3, U2 control. These data trends both have associated with them a very low probability (at most .001 that observed differences are sufficiently great to be attributable to instruction and are not chance occurrences.

The subtraction-without-regrouping test had the same format as the addition test. The test was in two parts: Part I had 20 problems and was given under power conditions; Part II had 10 problems and was given under speed conditions. The tests were scored on the basis of 30 possible points. In Table 11 (given on the following page) are presented means and adjusted means for the subtraction-without-regrouping total test scores.

Table 11

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for subtraction without regrouping total scores by groups and the significance level of the F-statistic for the 1 x 5 and 1 x 6 analyses of covariance with KeyMath scores and with KeyMath and IQ scores, respectively.

Group		1x5			1x6		
	$\bar{X}$	$A\bar{X}$	F	P	$A\bar{X}$	F	P
U1	28.20	28.52			28.81		
U2	28.60	28.78			29.12		
U3	29.00	29.14			29.79		
M	26.50	25.26			25.72		
2M	26.50	26.93			27.22		
C	29.80				28.12		
			.92	.999		.943	.999

These adjusted means order the groups as U3, U2, U1 (control), 2M, M. The associated probability that the observed differences are due to instruction is at most .001.

Again the writer suggests caution about the interpretation of these data trends because of the very small sample size.

Some summary remarks which reflect the investigator's interpretation of these data follow:

1. For developing in second graders skill and understanding for addition and subtraction of numbers named by 2-digit numerals the abacus, counting sticks, and Dienes blocks are equally effective as single manipulative aids.

2. There is in this data no evidence to suggest that the multiplicity of manipulative aids is an important variable for developing in second graders skill and understanding of addition and subtraction of numbers named by 2-digit numerals.

The children's responses in Interview 2 were coded into the six categories discussed on page 63. The number concept index derived from this coding (see page 63) is presented in Table 12; each index (4-tuple of numbers) indicates the number of correct responses in each of the 4 categories (see page 63) as a percent of the total number of attempts by groups.

Table 12

Summary of coded responses indicating the number of correct responses in Categories 1, 2, 3, and 4 as a percent of the total number of responses coded.

Group	Category 1	2	3	4
U1	3.2	14.9	3.2	26.6
U2	0	46.8	0	18.1
U3	0	40.4	4.3	31.9
M	0	19.2	5.1	44.9
2M	0	24.3	4.3	10.0
C	0	14.7	2.9	39.2
Exp H^+UH^-	0	50.8	3.4	15.8

It is of interest to compare the number concept indices presented in Table 12 with those presented in Table 7. The indices in Table 7 code the behaviors of children when they were working on tasks that dealt specifically with place value number situations. The indices in Table 12 present indices which reflect children's behavior when they were working on 2-digit addition and subtraction. Addition and subtraction of 2-digit numbers involves an application of place value concepts but the application is facilitated by a learned algorithm. It is of particular interest that a much higher percent of the children's responses suggested behavior which was coded in categories which suggest that the children interpreted 2-digit numbers as two separate entities.

A description of behavior which the investigator coded in Category 4 is as follows. Given the problem

$$\begin{array}{r} 46 \\ +\ 23 \\ \hline \end{array}$$

to work, the child might be heard to say "6 plus 3, 9; 4 plus 2, 6; sixty-nine." When asked to explain how the answer was gotten the explanation would be like "well, 6 and 3 is 9, 4 and 2 is 6." Additional queries from the interviewer would not bring the child to suggest that the 4 and 2 represent 40 and 20, respectively. The child's interpretation seemed to center on these as a 4 and a 2.

It is also interesting that almost no behaviors in Interview 2 were coded in Category 1. For example, for the problem

$$\begin{array}{r} 46 \\ +\ 26 \\ \hline \end{array}$$

almost no children were observed to say in explaining their process: "4 tens and 2 tens is 6 tens." A reading of the group teacher's logs gives some explanation for this. It is apparent from these logs that children find the thinking represented by 4 tens and 2 tens is 6 tens much more difficult than the thinking represented by 40 plus 20 is 60.

Quite a high percent of children's responses were coded in Category 2. This suggests that the investigator interpreted these behaviors to mean that the children were thinking of 2-digit numbers as a multiple of tens and ones. A description of behavior which the investigator coded in this category is as follows. Given the problem

$$\begin{array}{r} 46 \\ +\ 23 \\ \hline \end{array}$$

to work the child might be observed to say 6 plus 3, 9; forty plus twenty, sixty; sixty-nine. Or when asked to explain his process, the child would say "6 and 3 makes 9 (pointing) and forty and twenty is sixty (pointing at 4, 2 and 6)."

More than 40% of the observed behaviors for children in groups U2 and U3 were coded in Category 2, while all other groups had fewer than 25% coded in this category. The range of the percents of behaviors coded in Category 4 (2-digit numbers interpreted as 2 separate entities) ranged from a high of 44.9% in Group M to a low of 10% in Group 2M.

An interpretation of this data is that both children in the experimental groups and the control group exhibited a lower level of thinking concerning the concept of 2-digit numbers while doing addition and subtraction than when dealing with place value concepts per se. Maybe this is not surprising. Indeed, this may be exactly the number concept which the algorithm process for addition and subtraction encourages. Moreover, it may be the number interpretation which facilitates efficient algorithmic processing. On the other hand the H^+ UH $^-$ group exhibited behavior about 51% of which was coded in Category 2. The performance scores on addition and subtraction of this group is also higher. These are possibly coincidental happenings, not cause and effect. Indeed U2 and U3 experimental groups exhibited a higher percent of behaviors in Category 2 than did other experimental groups but their performance scores are not consistently higher than the other groups' scores.

The implication of this information to teaching is not clear. Some clarification might be obtained if systematic comparison was made of skill and understanding of 2-digit addition and subtraction of two groups of children after receiving instruction in one of two modes. One mode would emphasize the language 4 tens and 2 tens is 6 tens (or 40 plus 20 is 60), while the other concentrates on the least amount of information necessary to process the algorithm; that is, emphasizes addition through symbol manipulation.

EVALUATION INTERVIEW 3 - ADDITION AND SUBTRACTION FOR NUMBERS NAMED BY 2-DIGIT NUMERALS, WITH REGROUPING.

The third evaluation interview was concerned with an evaluation of the children's concepts of addition and subtraction for numbers named by 2-digit numerals in which the algorithm process required regrouping. Children were presented with situations which required them to display numbers with objects and to trade to display the same number but with more or fewer tens. Addition and subtraction problems involving 2-digit numerals were presented which required regrouping. Two of the items required the children to perform an addition problem and subtraction problem using a manipulative aid which was new to them. Two of the items presented children with a situation in which they were asked to judge whether or not a problem was worked correctly and if not, to explain why not.

The evaluation questions were determined in advance by the evaluators, according to the objectives of the written teaching materials. The questions and test materials were presented to the children according to a prewritten script. The complete script for the structured interview is contained in Appendix D.

Shown in Table 13 (given on the following 2 pages) is the profile sheet used to record the information about each child's performance and also the number and percent of the children in each group who were judged by the evaluators to have performed satisfactorily on each item.

Table 13

Student Profile Sheet for Interview 3 with number and percent of students who responded correctly to each item by groups

Item	Skill	Skill Score	Understanding	Understanding Score	U1 (5)[a] #	U1 %	U2 (5) #	U2 %	U3 (6) #	U3 %	M (4) #	M %	2M (4) #	2M %	C (5) #	C %
1	Read:	0 1			4	80	5	100	5	83.3	4	100	4	100	4	80
			Show:	0 1	4	80	4	80	5	83.3	4	100	4	100	5	100
			What no.:	0 1	4	80	4	80	2	33.3	4	100	2	50	4	80
			Regroup:	0 1 2	1*	80	0	0	1	16.7	0	0	1	25	1	20
					4	80	3	60	4	66.7	3	75	3	75	1	20
			What no.:	0 1	4	80	4	80	5	83.3	3	75	3	75	2	40
2	Read:	0 1			3	60	5	100	6	100	4	100	4	100	5	100
			Show:	0 1	5	100	5	100	6	100	4	100	4	100	5	100
			What no.:	0 1	5	100	5	100	5	83.3	4	100	4	100	4	80
			Regroup:	0 1 2	0	0	1	20	1	16.7	0	0	1		1	20
					5	100	4	80	5	83.3	4	100	3	75	2	40
			What no.:	0 1	4	80	4	80	5	83.3	3	75	2	50	3	60
3	Show addends:	0 1 2			0	0	0	0	2	33.3	0	0	1	25	3	60
					4	80	5	100	4	66.7	3	75	2	50	1	20
			Answer:	0 1 2			1	20	2	33.3	0	0	1	25	3	60
					2	40	4	80	4	66.7	4	100	2	50	5	100
4	Show minuend:	0 1			3	60	3	60	4	66.7	3	75	2	50	3	60
			Answer:	0 1 2	2	40	2	40	5	83.3	2	50	1	25	4	80
					2	40	2	40	1	16.7	2	50	2	50	0	0
5	Read:	0 1			5	100	5	100	6	100	4	100	4	100	4	80
	Answer:	0 1 2			1	20	0	0	0	0	2	50	0	0	0	0
					3	60	3	60	6	100	2	50	4	100	4	80
	Read answer:	0 1			5	100	5	100	6	100	4	100	4	100	5	100
			Explain:	0 1 2	1	20	0	0	0	0	2	50	2	50	0	0
					2	40	3	60	6	100	2	50	2	50	4	80
6	Read:	0 1			5	100	5	100	6	100	4	100	3	75	4	80
	Answer:	0 1 2			1	20	0	0	0	0	2	50	0	0	0	0
					3	60	2	40	4	66.7	2	50	3	75	5	100
	Read answer:	0 1			5	100	5	100	5	83.3	4	100	3	75	5	100
			Explain:	0 1 2	2	40	0	0	1	16.7	2	50	1	25	1	20
					2	40	2	40	3	50	2	50	2	50	3	60

(CONTINUED)

Table 13

(continued)

Item	Skill Score	Understanding Score	U1 (5)[a] #	U1 %	U2 (5) #	U2 %	U3 (6) #	U3 %	M (4) #	M %	2M (4) #	2M %	C (5) #	C %
7		Show addends: 0 1 2	0	0	1	20	0	0	1	25	0	0	0	0
			3	60	4	80	5	83.3	2	50	3	75	5	100
		Answer: 0 1 2	0	0	0	0	2	33.3	0	0	0	0	1	20
			4	80	3	60	2	33.3	3	75	2	50	3	60
8		Show minuend: 0 1	4	80	4	80	3	50	3	75	1	25	5	100
		Answer: 0 1 2	0	0	1	20	1	16.7	0	0	0	0	1	20
			4	80	2	40	1	16.7	2	50	1	25	2	40
9	Read: 0 1 2		3	60	4	80	2	33.3	1	25	2	50	1	20
			2	40	1	20	3	50	2	50	1	25	3	60
		Explain: 0 1 2	0	0	0	0	0	0	1	25	0	0	0	0
			4	80	1	20	5	83.3	2	50	2	50	2	40
10	Read: 0 1 2		0	0	0	0	0	0	0	0	0	0	0	0
			5	100	5	100	5	83.3	3	75	1	25	4	80
	Answer right: 0 1		3	60	2	40	1	16.7	2	50	1	25	3	60
		Explain: 0 1 2	0	0	1	20	1	16.7	0	0	0	0	2	40
			2	40	2	40	2	33.3	3	75	1	25	2	40

*The upper number indicates the number of children who received a 1, the lower number who received 2.

aThe number in parentheses indicates the number in the group.

In Table 14 are presented adjusted and unadjusted means for the five experimental groups for skill and understanding scores derived from Interview 3. The means were adjusted as described for Interview 1 scores.

Table 14

Means ($\overline{X}$) and adjusted means ($A\overline{X}$) for Interview 3 skill and understanding scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ scores

Group	Skill				Understanding			
	$\overline{X}$	$A\overline{X}$	F	P	$\overline{X}$	$A\overline{X}$	F	P
U1	11.80	12.98			21.20	23.75		
U2	11.50	14.10			21.60	22.48		
U3	12.17	13.29			24.00	24.57		
M	11.75	11.04			25.25	17.08		
2M	10.50	11.68			17.25	20.28		
			.506	.999			.817	.999

The adjusted means for the skill scores order the groups from highest to lowest performance in the order U2, U3, U1, 2M, M. For the understanding scores the adjusted means order the groups as U3, U1, U2, 2M, M. Also presented in Table 14 is the F-statistic and the significance level of the F from the analysis of covariance which was conducted to determine whether there are statistically significant differences between any of the adjusted group means. The analysis was conducted using KeyMath and IQ test scores as covariates. This information suggests that the data tre. apparent for the skill scores has associated with it a probability of at most .001 that any observed differences are due to instructional effects rather than being a chance occurrence. The date trend for the understanding scores has a similarly low probability.

Similar information concerning the analysis of data including the control group is given in Table 15 (shown on the following page). The adjusted mean skill and understanding scores respectively order the groups as U2, U3, U1, 2M, M, Control and as U3, U1, U2, 2M, M, Control.

For the understanding scores there is an associated probability of .95 that observed differences are due to instruction. It is apparent that the control group did significantly poorer on the understanding of addition and subtraction involving regrouping although their skill scores did not differ significantly, based on adjusted mean scores.

Table 15

Means ($\bar{X}$) and Adjusted Means ($A\bar{X}$) for Interview 3 Skill and Understanding Scores by Groups and the Significance Level of the F-Statistic for the 1 x 6 analysis of Variance with KeyMath Scores

Group	Skill				Understanding			
	$\bar{X}$	$A\bar{X}$	F	P	$\bar{X}$	$A\bar{X}$	F	P
U1	11.80	13.23			21.20	25.37		
U2	14.00	15.21			21.60	25.14		
U3	13.33	14.47			24.00	27.33		
M	13.75	11.94			25.25	19.96		
2M	10.50	12.18			17.25	22.17		
C	15.20	11.29			19.80	8.39		
			.833	.999			2.650	.05

In addition to the interview evaluation conducted by the evaluators, the investigator, in conjunction with the group teachers, prepared paper and pencil tests to measure children's computational skill. Separate tests were written for addition and subtraction. The addition-with-regrouping test was in three parts. The first part had twenty addition items and was given under power conditions. Part II had 10 addition items and was given as a timed test. Part III had four addition items; the children were given the opportunity to use manipulatives on this part which was given under power conditions. Thus the total test was scored on a basis of 34 points. In Table 16 (shown on the following page) are presented means and adjusted means for the addition-without-regrouping total test scores.

Table 16

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for addition-with-regrouping total test scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ scores

Group	$\bar{X}$	$A\bar{X}$	F	P
U1	27.80	27.68		
U2	24.00	25.30		
U3	31.33	32.67		
M	27.75	23.41		
2M	33.25	34.00		
			4.06	.017

Adjusted means for the 1 x 5 analysis order the groups from highest to lowest performance in the order 2M, U3, U1, U2, M. The associated probability that this trend is other than a chance occurrence is .98. Moreover, the data suggest that this is due, at least in part, to the fact that the adjusted mean for group 2M is very much greater than the adjusted mean for M, for example. However, due to the fact that the mean time for group 2M was also very great in comparison to the other groups, it is more likely that this difference in mean achievement scores is due to the time used rather than because of achievement differences due necessarily to instruction. Thus, caution about interpretation of the data due to the very small sample size is again suggested.

The subtraction-with-regrouping was similar in all respects to the addition-with-regrouping. In Table 17 (shown on the following page) are presented means and adjusted means for the subtraction-without-regrouping test.

Adjusted means for this 1 x 5 analysis order the groups from highest to lowest performance in the order 2M, U3, U2, U1, M. The associated probability that this trend is not a chance occurrence is at most .001. Thus, there is little reason to believe that observed performance differences on this subtraction test were due to differences in achievement caused by instructional variation. While the adjusted mean for the group 2M is the greatest, the mean time for the 2M group was also higher than the mean time for the other groups.

Table 17

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for subtraction-with-regrouping total test scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ scores

Group	$\bar{X}$	$A\bar{X}$	F	P
U1	23.40	24.47		
U2	23.80	24.49		
U3	28.00	28.56		
M	27.50	22.93		
2M	33.75	35.26		
			.739	.999

The following summary remarks concerning the data for evaluation of children's performances on addition and subtraction of numbers named by 2-digit numerals is offered by the investigator.

1. For developing in second graders skill and understanding for addition and subtraction of numbers named by 2-digit numerals the abacus, counting sticks, and Dienes blocks are equally effective as single manipulative aids.

2. There is in this data no evidence to suggest that the multiplicity of types of manipulative aids is an important variable for developing in second graders skill and understanding of addition and subtraction of numbers named by 2-digit numerals for which the algorithm requires regrouping.

3. There is evidence in these data to suggest that (See Tabl 15) second grade children who use manipulative aids in learning concepts of addition and subtraction of numbers named by 2-digit numerals are better able to display understanding of these concepts.

The children's responses to questions asked in Interview 3 were not coded into the six categories discussed on page 63. Although an attempt was made

to do so, the questions posed in this interview were unfortunately not of the type to evoke categorizable responses. (Elaboration upon this is given in the Discussion Section of this report.)

EVALUATION INTERVIEW 4, 3-DIGIT NUMERATION

The fourth and final interview was concerned with an evaluation of the children's concepts of 3-digit numeration. Also, it was concerned with further evaluation of their understanding of place value numeration, addition, and subtraction, and the regrouping process as exhibited by their ability to transfer and extend the algorithm for addition and subtraction with regrouping for 2-digit numerals to 3-digit numerals. Children were presented with situations which required them to display 3-digit numbers with objects and to regroup the objects to more or fewer hundreds or tens, while keeping the total number the same. Addition and subtraction problems involving 3-digit numerals which required regrouping were presented, and children were asked to solve these with familiar and unfamiliar manipulations and also symbolically without the aid of manipulatives. Also, two problems involving the concepts of more and less were given in the context of 3-digit numerals

The evaluation questions were determined in advance by the evaluators, according to the objectives of the written teaching materials and the objective of evaluating for transfer of learning. The questions and test materials were presented to the children according to a pre-written script. The complete script for this structured interview is contained in Appendix E.

Shown in Table 18 (shown on the following 3 pages) is the profile sheet used to record the information about each child's performance and also the number and percent of the children in each group who were judged by the evaluators to have performed satisfactorily on each item.

Table 18

Student Profile Sheet for Interview 4 with the number and percent of students who responded correctly to each item by groups

Item	Skill	Score	Understanding	Score	Group U1	Group U2	Group U3	Group M	Group 2M	Group C
1	Read	0 1			5	4	6	4	4	5
			Show	0 1	5	5	6	4	3	5
			What no.	0 1	2	2	2	3	2	1
			Regroup	0 1 2	0 4	0 3	0 6	0 3	0 4	0 3
			What no.	0 1	4	3	6	3	4	2
2	Read	0 1			5	5	6	4	4	5
			Show	0 1	5	5	6	4	4	5
			What no.	0 1	5	5	6	4	4	5
			Regroup	0 1 2	0 4	0 3	0 4	1 3	0 4	2 2
			What no.	0 1	4	2	6	4	3	4
3	Show addends	0 1 2			1 4	1 4	2 4	0 3	0 2	0 4
			Answer	0 1 2	1 2	0 3	2 4	1 2	0 0	0 2
4	Show addends	0 1 2			1 4	1 4	1 4	0 2	1 0	0 3
			Answer	0 1 2	3 0	1 2	1 3	0 3	0 1	2 2
5	Show minuend	0 1			5	5	5	2	2	3
			Answer	0 1 2	2 0	0 3	1 4	0 2	0 2	1 1
6	Show minuend	0 1			4	5	3	2	2	4
			Answer	0 1 2	2 1	1 2	0 2	1 0	1 0	2 1

Table 18
(continued)

Item	Skill Score		Understanding Score		Group U1	U2	U3	M	2M	C
7	Read	0 1			4	5	5	4	2	5
	Answer	0 1 2			0 1	2 1	1 2	0 2	1 2	0 5
	Read answer,	0 1			3	5	2	3	3	5
			Explain	0 1 2	2 1	1 2	2 2	2 1	2 1	0 5
8	Read	0 1			4	5	5	4	1	5
	Answer	0 1 2			1 0	0 1	2 2	1 1	3 0	0 2
	Read answer	0 1			1	4	4	4	3	5
			Explain	0 1 2	3 0	2 1	2 2	2 1	0 2	1 2
9			Show addends	0 1 2	0 4	1 4	3 2	0 2	0 2	0 4
			Answer	0 1 2	0 3	1 2	0 2	1 2	1 1	1 3
10			Show minuend	0 1	4	3	3	2	2	4
			Answer	0 1 2	0 1	0 3	0 2	1 1	0 2	2 1
11	Right	0 1			1	1	3	4	2	2
			Explain	0 1 2	2 0	1 0	1 2	1 2	0 0	0 1
12	Right	0 1			0	3	2	2	1	4
			Explain	0 1 2	1 0	1 2	1 2	0 2	0 0	2 2
13	Answer	0 1			4	2	5	3	3	1
			Explain	0 1 2	1 3	1 0	1 3	0 3	0 1	1 1

Table 18

(continued)

Item	Skill Score	Understanding Score	U1		U2		U3		M		2M		C	
14	Answer 0 1		3		2		4		3		1		1	
		Explain 0 1 2	0	3	0	1	0	3	0	3	0	1	2	1
15	Answer 0 1 2		1	4	1	3	3	3	1	3	2	0	1	1

In Table 19 (shown below) are presented adjusted and unadjusted means for the five experimental groups for skill and understanding scores derived from Interview 4. The means were adjusted as described for Interview 1 scores.

Table 19

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for Interview 4 skill and understanding scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and I.Q. Scores.

Group	Skill				Understanding			
	$\bar{X}$	$A\bar{X}$	F	P	$\bar{X}$	$A\bar{X}$	F	P
U1	14.60	15.08			22.40	23.17		
U2	14.00	14.14			19.00	20.17		
U3	15.33	15.40			19.83	20.93		
M	15.25	13.19			22.00	16.35		
2M	14.00	14.61			17.75	19.34		
			1.157	.364			.958	.999

The adjusted means for the skill scores order the groups from highest to lowest performance in the order U3, U1, 2M, U2, M. For the understanding scores the adjusted means order the groups as U1, U3, U2, 2M, M. Also presented in Table 19 is the F-statistic and the significance of the F from the analysis of covariance which was conducted to determine whether there are statistically significant differences between any of the adjusted group means. The analysis was conducted using KeyMath and IQ scores as covariates. This information suggests that the data trend for the skill scores has associated with it a probability of only .64 that observed differences are other than a chance occurrence. The data trend for the understanding scores has a probability of at most .001 that the differences are other than a chance occurrence.

Similar data concerning the analysis of data including the control group is given in Table 20 (below). The data indicate that the control group performed lower than any of the experimental groups. Moreover, these data trends have associated probabilities of .68 and .96 for skill and understanding, respectively, that the observed differences are not due to chance.

Table 20

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for Interview 4 skill and understanding scores by groups and the significance level of the F-statistic and the 1 x 6 analysis of variance with KeyMath scores.

Group	Skill				Understanding			
	$\bar{X}$	$A\bar{X}$	F	P	$\bar{X}$	$A\bar{X}$	F	P
U1	14.60	15.63			22.40	25.11		
U2	14.00	14.87			19.00	21.30		
U3	15.33	16.16			19.83	21.99		
M	15.25	13.94			22.00	18.56		
2M	14.00	15.21			17.75	20.94		
C	14.80	11.98			18.60	11.19		
			1.244	.322			2.869	.038

In addition to the interview evaluation conducted by the evaluators, the investigator, in conjunction with the group teachers, prepared a paper and pencil test to measure the children's ability to perform various place value tasks. The place value test was written and administered in seven parts. Part I of the test presented children with 12 pictures-of-manipulatives displays for numbers named by 2- and 3-digit numerals, and the children were required to write the corresponding numeral. Part II of the test presented the children with 4 picture-of-manipulative displays. The teacher wrote the corresponding numeral on the chalkboard, circled the hundreds, tens, or units digit, and the children were required to circle the corresponding part of the picture. Part III presented the children with 6 picture-of-manipulative displays for 3-digit numbers. The children were required to circle one of our numerals to indicate the number shown. Part IV of the test presented the children with 4 picture-of-manipulative displays showing 4 hundreds, 7 tens, and 7 ones. A number less than 477 was then written by the teacher on a chalkboard. The children were required to color the pictures of objects to indicate the number. In Part V of the test children were presented with 8 numerals written in a place value chart and were required to write the numeral without the chart. Part VI oresented the children with 6 written statements like

____ hundreds

____ ones

____ tens

and the children were required to write the corresponding numeral. In Part VII the children were presented with six 3-digit numerals and were required to write the number which is 1 or 10 more or less than the given number.

In Table 21 (shown on the following page) are presented means and adjusted means for the place value test. Adjusted means for the 1 x 5 and 1 x 6 analyses order the groups from highest to lowest performance in the order 2M, U3, U1, U2, M (control). The data trend for the 1 x 5 analysis has associated with it a probability of at most .001 that any difference between the experimental groups are other than a chance occurrence. However, the data trend for the 1 x 6 analysis has an associated probability of about .99 that the differences are other than a chance occurrence, i.e., possibly an instructional difference. It is apparent that the difference is between the control group and each of the experimental groups.

Table 21

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for place value test total scores, by groups and the significance level of the F-statistic for the 1 x 5 and 1 x 6 analyses of covariance with KeyMath scores and with KeyMath and IQ scores and KeyMath scores, respectively.

	1 x 5				1 x 6			
Group	$\bar{X}$	$A\bar{X}$	F	P	$\bar{X}$	$A\bar{X}$	F	P
U1	40.40	41.64			40.40	42.46		
U2	41.60	41.67			41.60	43.34		
U3	41.33	41.25			41.33	42.97		
M	41.50	38.81			41.50	38.90		
2M	42.25	43.46			42.25	44.67		
C					33.20	27.59		
			.587	.999			3.807	.012

Some summary remarks which reflect the investigator's interpretation of these data follow:

1. For extending second grader's skill and understanding of numeration to 3-digit numerals, the abacus, counting sticks and Dienes blocks are equally effective as single manipulative aids.

2. There is no evidence to suggest that the multiplicity of manipulative aids is an important variable for extending second grader's skill and understanding of numeration to 3-digit numeration.

3. For developing in second graders skill and understanding to transfer knowledge of addition and subtraction algorithmic processes from 2-digit to 3-digit numerals, the abacus, counting sticks, and Dienes blocks are equally effective as single manipulative aids.

4. There is no evidence to suggest that the multiplicity of manipulative aids is an important variable for developing in second graders the skill and understanding to transfer

knowledge of addition and subtraction algorithmic processes from 2- to 3-digit numerals.

5. There is evidence to suggest that the systematic use of one or more manipulative aids increases second graders' skill and understanding of numeration and of addition and subtraction algorithms beyond children who are taught in a mode in which manipulatives are not a systematic part of the instructional process; i.e., a traditional approach in which the instructional process is more dependent on a contemporary printed textbook.

The children's responses in Interview 4 were coded into six categories discussed on page 63. The number concept indices derived from this coding (see page 63) is presented in Table 22 (given below). The number concept index derived from this coding (see page 63) indicates the number of correct responses in each of 5 categories (see page 63) as a percent of the total number of attempts by groups.

Table 22

Summary of coded responses indicating the number of correct responses in categories 1, 2, 3, 4, and 5 as a percent of the total number of responses coded.

	Category				
Group	1	2	3	4	5
U1	8.5	26.8	4.2	5.6	7.0
U2	7.0	32.4	7.0	4.2	8.6
U3	6.8	26.2	3.9	8.7	10.7
M	9.7	37.1	0	0	24.2
2M	0	34.5	0	8.6	1.7
C	0	28.3	2.7	13.5	8.1
Exp $H^+ \cup H^-$	10.9	43.7	4.6	5.7	1[illegible].5

It is of interest to compare the number concept indices presented in Tables 5, 12, and 22. While the percent of responses for all groups coded in category 4 showed a marked increase in Interview 2 over Interview 1, the percent in this category again dropped in Interview 4. This is probably due to

the type of tasks in the various interviews. The first interview dealt almost exclusively with place value tasks, Interview 2 with addition and subtraction without regrouping, and Interview 4 was a combination of place value tasks and the application of place value concepts to addition and subtraction in which regrouping was required. One might conjecture that the regrouping activity elicits a behavior which suggests a "higher" number concept level than addition and subtraction without regrouping. For problems that require no regrouping, it might be possible to function quite adequately by thinking about the digits of a multi-digit numeral as separate entities. However, the regrouping forces the thinking to extend to a relationship between the digits of the numeral, or at least a more accurate meaning of each digit in the context of a multi-digit numeral. It is not clear from the data whether or not there is a cause and effect relationship between level of number concept and ability to perform addition and subtraction problems. That is, we cannot, from these data, necessarily conclude that success with the addition-with-regrouping algorithm is facilitated by a "higher" number concept index. It does, however, seem reasonable to conjecture that children who exhibit behavior suggestive of a "higher" number concept are more likely to have internalized the concept of "ten-is-a-unit" idea. That is, that 10 ones becomes a unit of 1 ten, that 10 tens becomes a unit of 1 hundred, etc. This "ten-is-a-unit" idea seems essential for an adequate understanding of place value numeration, the regrouping process and the application of these concepts to addition and subtraction. Thus, to argue that a "higher" number concept will facilitate performance on addition and subtraction with regrouping appears plausible.

Another observation from Table 22, is that the percent (about 70%) of responses for the group Exp $H^+ \cup H^-$ categorized in Categories 1, 2, and 5, was almost twice the percent (about 36%) for the control group. Recalling that the Exp $H^+ \cup H^-$ and control groups were comparable on pretest scores of the KeyMath one seems justified to conclude (or at least conjecture) that the systematic use of manipulative aids causes second grade children to exhibit behaviors on place value tasks and on addition and subtraction with regrouping tasks which are suggestive of a "higher" level of place value concept formation. This, together with the fact that the experimental groups overall (including many very less capable children than the control group) performed better on Interview 4 tasks and on the investigator-written numeration test than did the control group, suggests that this "higher" level number concept has a facilitating effect on problems involving the application of place value concepts. One must, however, be cautious about this conclusion because the data derived to indicate the number concept level comes from the same source as the data derived to indicate success of number concept application.

ADDITIONAL EVALUATIONS

To determine whether there were differences among the groups on the retention of skills learned earlier in the year, an investigator-written retention test was administered. Each of the three parts consisted of twelve items and was administered as a power test. Each part of the test included basic fact problems, problems without regrouping and problems with regrouping. Part I, II, and III had only addition, only subtraction, and both addition and subtraction items, respectively. This test was administered to the experimental groups and the control group.

In Table 23 (given below) are presented means and adjusted means for the Retention Test for the 1 x 5 and 1 x 6 analyses.

Table 23

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for Retention Test total scores by groups and the significance level of the F-statistic for the 1 x 5 and 1 x 6 analysis of covariance with KeyMath and IQ and with KeyMath scores, respectively.

Group	1 x 5				1 x 6			
	$\bar{X}$	$A\bar{X}$	F	P	$\bar{X}$	$A\bar{X}$	F	P
U1	28.40	29.64				29.77		
U2	28.00	28.03				29.16		
U3	31.66	31.53				32.76		
M	29.75	27.19				28.01		
2M*	32.25	26.58				33.87		
C					30.00	26.25		
			.748	.999			.905	.999

These adjusted means order the groups from highest to lowest in performance as 2M, U3, U1, U2, M, (control). The data trends for both the 1 x 5 and 1 x 6 analyses have associated probabilities of at most [illegible] that observed differences are due to other than a chance occurrence.

An "End of the Year Math Test" was written and administered by the regular teacher of the class in which children of the experimental groups were enrolled. This test included items on counting sets of dots, basic addition and subtraction facts, addition and subtraction with and without regrouping for 2- and 3-digit numerals, and counting by 1's, 2's, 5's, 10's and 100's. The test was scored on a basis of 100 points by the regular teacher. The adjusted and unadjusted means of these scores are presented in Table 24 (given below). These adjusted means order the groups from highest to lowest performance as U3, 2M, U1, U2, M.

Table 24

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for the regular teacher's End-of-Year Math Test total scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ scores.

Group	$\bar{X}$	$A\bar{X}$	F	P
U1	72.20	77.99		
U2	64.40	65.29		
U3	79.60	80.31		
M	72.50	57.09		
2M	72.25	78.39		
			1.060	.407

This data trend has an associated probability of only .59 that the observed differences are other than a chance occurrence.

The Comprehensive Test of Basic Skills test was administered to the children in the experimental groups as part of the school district's regular testing program. The scores obtained by the experimental group children were made available by groups to the investigator. In Tables 25, 26, and 27 (given on the following two pages) are presented adjusted and unadjusted means for CTBS comprehension, application and total scores, respectively. The adjusted means for the CTBS comprehension scores order the groups from highest to lowest performance as U3, U2, M, 2M, U1. These adjusted means are very close and have a probability of at most .001 that the differences are other than a chance occurrence. The adjusted mean application scores order the groups as U1, U3, U2, 2M, M with an associated probability of .82 that the differences are other than a chance occurrence. The adjusted means for the CTBS total

scores order the groups as U3, U1, U, 2M, M, and this data trend has an associated probability of at most .001 that the differences are other than chance occurrence.

Table 25

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for the CTBS comprehension test scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ score.

Group	$\bar{X}$	$A\bar{X}$	F	P
U1	14.80	15.62		
U2	16.00	16.69		
U3	18.00	17.84		
M	20.75	16.53		
2M	15.00	16.30		
			.240	.999

Table 26

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for the CTBS application test scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ scores.

Group	$\bar{X}$	$A\bar{X}$	F	P
U1	17.80	19.27		
U2	16.20	16.45		
U3	16.67	16.33		
M	17.45	13.12		
2M	14.00	15.63		
			1.674	.202

Table 27

Means ($\overline{X}$) and adjusted means ($A\overline{X}$) for CTBS total test scores by groups and the significance level of the F-statistic for the 1 x 5 analysis of covariance with KeyMath and IQ scores.

Group	$\overline{X}$	$A\overline{X}$	P	P
U1	32.60	34.87		
U2	32.20	33.30		
U3	34.67	35.49		
M	38.00	29.60		
2M	29.00	31.93		
			.780	.999

Each of the 4 interviews had items which required the children to display their skill and understanding of concepts by using a manipulative aid which was new to them--that is, one which had not been used in any of the teaching groups. Although it was believed that these "new manipulatives" would not give a testing advantage to any of the groups, they were classifiable into the same category of embodiment as Dienes blocks; that is, 1 hundred and 1 ten were not decomposable into 10 tens and 10 ones, respectively.

Means and adjusted means derived from the 1 x 6 analysis of covariance with KeyMath total scores are given in Table 28 (given on the following page). These adjusted means order the groups from highest to lowest performance as U2, U1, U3, 2M, M. This data trend has an associated probability of 0.84 that some observed difference(s) are due to an instructional difference.

It is interesting to observe that according to the adjusted means, each of the three teaching groups in which children's learning depended on a single embodiment outperformed those groups which used several embodiments. In some sense one might believe that the ability to demonstrate skill and understanding with a new embodiment reflects the extent to which the children have abstracted the concept(s). This is contrary to expectation; however, to the extent that we can have faith in the statistics due to the small sample size, it does appear doubtful that children having had experience with learning from several embodiments can demonstrate abstraction of the concept more readily than

children learning from a single embodiment. It is emphasized that this observation is very tenuous and requires further investigation.

Table 28

Means ($\bar{X}$) and adjusted means ($A\bar{X}$) for items involving a new manipulative from Interviews 1, 2, 3, and 4, by groups with KeyMath scores.

Group	$\bar{X}$	$A\bar{X}$	F	F
U1	17.80	21.68		
U2	19.00	22.29		
U3	14.81	17.92		
M	19.25	14.35		
2M	11.75	16.31		
C	23.00	12.42		
			1.780	0.158

Shown in Table 29 (given on the following page) are the results of a 1 x 6 analysis of covariance to determine which achievement differences exist between the Exp H^+ U H^- group and the control group on the several evaluations.

The split-half reliability for selected investigator written tests are given in Table 30 (given below).

Table 30

Split-half reliability coefficients for selected investigator written tests.

TEST	RELIABILITY
Addition without regrouping	0.80
Subtraction without regrouping	0.84
Addition with regrouping	0.92
Subtraction with regrouping	0.98
Retention test	0.91

Table 29

Means ($\bar{X}$), adjusted means ($A\bar{X}$) and F and P Values for Exp $H^{+} \cup H^{-}$ and Control for each evaluation.

Evaluation	Group					
	$H^{+} \cup H^{-}$		C			
	$\bar{X}$	$A\bar{X}$	$\bar{X}$	$A\bar{X}$	F	P
Interview I S	18.50	18.99	16.80	16.03	7.925	0.009
U	11.25	11.78	8.30	7.55	7.215	0.012
Interview II S	35.37	35.80	31.60	30.90	1.643	0.241
U	32.38	31.36	21.80	23.43	4.196	0.047
Add W/O regrouping	28.37	28.41	29.20	28.24	1.087	0.376
Subt. W/O regrouping	29.75	29.80	29.80	29.72	1.263	0.325
Interview III S	17.25	17.43	15.20	14.91	1.811	0.212
U	33.13	33.55	19.80	19.12	6.886	0.013
Interview III S	17.12	14.80	17.37	14.30	2.827	0.105
U	26.37	26.66	18.50	18.13	6.384	0.016
Numeration	44.87	33.20	45.88	31.58	5.978	0.002
Retention	32.88	32.98	30.00	29.74	0.749	0.999

[1] N=8 and 5, respectively for Exp $H^{+} \cup H^{-}$ and Control.

VI. SUMMARY, CONCLUSIONS, DISCUSSION AND RECOMMENDATIONS

SUMMARY AND CONCLUSIONS

The teaching experiment reported herein had as its major purpose the collection and interpretation of essentially qualitative data. The objectives were to identify important variables concerning teaching and learning through the use of manipulative aids, and to state certain hypotheses concerning these variables.

The experiment was conducted in a public elementary school in a southeastern city of 86,000. The area served by the school includes many families of low socioeconomic status and, also, the married student housing complex of a large university. The duration of the experiment was most of a school year--mid-October to the end of the school year.

Initially, the 30 children in an intact second grade class were assigned to one of five teaching groups according to a stratified random procedure. This resulted in 5 groups of 6 children which were comparable in mathematical ability as measured by the KeyMath Diagnostic Test. After attrition, the experimental groups had 5, 5, 6, 4 and 4 children. The experimental groups were identified according to the embodiment which served as the learning aid for the group: U1--counting sticks, U2--Dienes blocks, U3--abacus, M--counting sticks, Dienes blocks, and abacus, 2M-- counting sticks and unifix cubes, Dienes blocks and grid paper, abacus and counting chips.

Teaching materials were developed for the experiment so that teaching was comparable across the five experimental groups except for the embodiment used for the group. The development of the materials represented a standard sequence for topics related to place value, addition, and subtraction and order concepts. The lessons were developed in groups according to the following topics:

1. 2-digit numeration.
2. Addition without regrouping
3. Subtraction without regrouping.
4. Addition with regrouping.
5. Subtraction with regrouping.
6. 3-digit numeration.

Evaluation of student progress was accomplished in two ways: (1) Two PMDC principal investigators, other than the investigator for this experiment, conducted evaluation interviews with the children; (2) Paper and pencil tests written by the investigator or the regular class teacher and also standardized tests were administered.

Although a number of covariance analyses of the data using SPSS Program ANOVA were conducted, the emphasis of the data analysis was on observation of data trends and "data snooping."

In Table 31 (given on the following page) are presented the orderings, by adjusted means, of the experimental and control groups from highest to lowest performance on each of the evaluations. Also reported is a probability which gives an indication about whether or not there is a difference between two or more group means attributable to teaching effects than to a chance occurrence.

Due to the small sample sizes in the treatment groups considerable caution must be exercised in the interpretation of the trends. The following statements are offered by the investigator as plausible conjectures based on the data.

When used as single embodiments, counting sticks Dienes blocks, and an abacus are, for second graders:

1. Not equally effective for developing the concept of 2-digit numerations, the abacus is less effective.

2. Equally effective for developing skill and understanding for addition and subtraction of numbers named by 2-digit numerals for which no regrouping is required.

3. Equally effective for developing skill and understanding for addition and subtraction of numbers named by 2-digit numerals for which regrouping is required.

4. Equally effective for extending the concept of numeration to 3-digit numerals.

Multiplicity of embodiments, in and of itself, provides a learning environment which when compared to learning environments defined by a single embodiment is, for second graders:

1. Superior for the purpose of developing the concept of 2-digit numeration.

2. Not superior for developing skill and understanding of addition and subtraction of numbers named by 2-digit numerals for which no regrouping is required.

Table 31

Ordering of experimental and control groups from highest to lowest performance by adjusted group means and the probability (P) that there exists a difference between two adjusted means which is not a chance occurrence.

Evaluation	Ordering of groups	P
Interview I		
Skill	2M, U2, M, U1, U3 (C)*	.91 (.93)
Understanding	2M, U1, U2, U3, M (C)	.60 (.83)
Interview 2		
Skill	U1, U2, 2M, U3, (C), M	.77 (.66)
Understanding	U2, U1, U3, 2M, M, (C)	.001 (.001)
Addition W/O regrouping	U1, 2M, M (C), U3, U2	.001 (.001)
Subtraction W/O regrouping	U3, U2, U1, (C), 2M, M	.001 (.001)
Interview 3		
Skill	U2, U3, U1, 2M, (C), M	.001 (.001)
Understanding	U3, U1, U2, 2M, M, (C)	.001 (.95)
Addition W/O regrouping	2M, U3, U1, U2, M	.98
Subtraction W regrouping	2M, U3, U2, U1, M	.001
Interview 4		
Skill	U3, U1, 2M, U2, M, (C)	.64 (.68)
Understanding	U1, U3, U2, 2M, M, (C)	.001 (.96)
Place Value	2M, U2, U1, U3, M, (C)	.001 (.98)
Retention	U3, U1, U2, M, 2M, (C)	.001 (.001)
Teacher Test	U3, 2M, U1, U2, M	.59
CTBS Comp.	U3, U2, M, 2M, U1	.001

* The ordering of the experimental groups is based on a 1 x 5 analysis, the ordering of the control group is based on the 1 x 6 analysis, probabilities in parentheses correspond to the 1 x 6 analysis.

(continued from page 99)

3. Not superior for developing skill and understanding of addition and subtraction of numbers named by 2-digit numerals for which regrouping is required.

4. Not superior for developing skill and understanding of addition and subtraction algorithmic processes from 2- to 3-digit numerals.

A learning environment defined by a systematic use of one or more manipulative aids compared to a learning environment in which this is absent is for second grade children:

1. More effective for developing 2-digit numeration.

2. More effective for developing the skill and understanding of addition and subtraction of 2-digit numbers when no regrouping is required.

3. More effective for developing the skill and understanding of addition and subtraction of 2-digit numbers when regrouping is required.
4. More effective for extending skill and understanding concerning 2-digit numeration to 3-digit numerals.

In three of the four interviews the children's responses were coded by the investigator into response categories representing a "level of numbers concept" suggested by that behavior. There were six levels of behavior identified which suggested one of the following interpretations of 2-digit (and 3-digit) numerals.

1. As tens and ones; for example, 24 as 2 tens and 4 ones. Similarly for 3-digit numerals as hundreds, tens, and ones.

2. As a multiple of ten and ones; for example, 24 as 20 and 4 ones. Similarly for 3-digit numerals: as the sum of multiples of one hundred and of ten and ones.

3. As ones; for example, 24 as 24 ones.

4. As separate entities; for example, 24 as a 2 and a 4.

5. As tens and ones, with ones greater than 9; similarly for 3-digit numerals--as hundreds, tens and ones with the number of tens and ones greater than nine.

6. Other; that is, responses that were not classifiable into any one of the other categories.

A summary of this coded data giving a number concept index for the groups is given in Table 32 (given on the following page). The numbers indicate the mean number of correct responses as a percent of the total number of attempts for each group.

Some observations about the collection and interpretation of these data follow. The coding of these responses was a <u>post hoc</u> type of investigation. That is, neither the experiment nor the evaluations were particularly designed to generate this data. The coding of this data occurred as a result of observing the responses of children to place value and addition and subtraction tasks. It does appear that this might prove to be a useful criterion concerning the evaluation of children's development of arithmetic concepts. The design of experiments was especially concerned with: (1) careful design of tasks so that children clearly exhibit behavior to suggest a number concept index; (2) investigation of whether certain methods of instruction are more effective in developing a "high" number concept index; (3) theoretical work to develop a means to indicate a number concept index with appropriate psychometric properties so that reliable comparison can be made between number concept indices; and (4) the relationship between number concept index and skill and understanding on other arithmetical tasks and concepts. In short, while the notion of identifying such a number concept index for children appears to have potential as a indicator of mathematical achievement, considerable refinement of the idea beyond the naive notion presented herein is necessary.

Another post hoc analysis conducted on the data derived from Interview 1, related to the questions of the relative success for high, middle and low ability children when response modes were oral, manipulative, or written. Again this analysis was not part of the experimental design but was suggested instead from an information observation that low ability students seemed to perform nearly as well during the enactive (manipulative) phase of instruction as did the high ability children. While an analysis of data did not support a statement as strong as this, it is apparent that, relative to high ability students, low ability students enjoy a much higher degree of success when the response made is manipulative than when the response made is symbolic. This information bears upon teaching practices in elementary schools. However, information available herein is, of course, tenuous and needs further experimental support. In particular, it is suggested that research be conducted to give indications about the relative success of high, middle, and low ability students when stimulus and response modes are presented to children within all sixteen cells of a 4 x 4 matrix having oral manipulative, pictorial, and symbolic modes of stimuli and responses, respectively, on each of the two dimensions of the matrix. Also, research should be addressed to identification of factors, in addition to mathematical achievement, which concern the questions of which of the sixteen stimulus-response modes best enables a particular child to communicate his understanding of a mathematical concept. We need to keep in mind that some children's apparent nonsuccess at a given mathematical task might be related to a lack of understanding of the task or an inability to communicate; the appropriate stimulus-response mode might facilitate both.

Table 32

Summary of number concept indices for Interviews 1, 2, and 4, as a percent of number of responses coded

Group	Interview	Category 1	2	3	4	5
U1	1	30.4	7.3	10.1	2.9	
	2	3.2	14.9	3.2	26.6	
	4	8.5	26.8	4.2	5.6	7.0
U2	1	40.8	16.3	4.1	6.1	
	2	0	46.8	0	18.1	
	4	7.0	32.4	7.0	4.2	8.6
U3	1	18.3	16.9	9.1	2.6	
	2	0	40.4	4.3	31.9	
	4	6.8	26.2	3.9	8.7	10.7
M	1	44.7	10.6	12.8	4.3	
	2	0	19.2	5.1	14.9	
	4	9.7	37.1	0	0	24.2
2M	1	39.6	3.8	1.9	0	
	2	0	24.3	4.3	10.0	
	4	0	34.5	0	8.6	1.7
C	1	27.1	11.9	18.6	10.1	
	2	0	14.7	2.9	39.2	
	4	0	28.3	2.7	13.5	8.1
Exp H^+UH^-	1	50.5	19.4	9.7	1.01	
	2	0	50.8	3.4	15.8	
	4	10.9	43.7	4.6	5.7	15.5

DISCUSSION AND RECOMMENDATIONS

As indicated by the brief review of related research as well as the results of this study, questions about how the use of manipulative aids affect the teaching and learning of mathematics have not received definitive answers. As a result of investigation of the data produced in this teaching experiment together with information obtained from many hours of observation of children and teachers working systematically with manipulative aids, at least the following is clear:

Whereas, it is apparent that the use of manipulative aids does facilitate children's learning of mathematics, their use does not by any means constitute a panacea for overcoming difficulties in the learning of mathematics, nor are procedures for the effective use of manipulatives well defined. Moreover, the question of whether or not teachers and children use manipulative aids in the teaching and learning of mathematics is not to be confused with the question of whether the learning taking place is rote or meaningful. It is apparent that children can and do engage in rote manipulation of objects as well as rote "manipulation" of symbols. The chances for meaningful learning are increased through the systematic use of manipulatives, but manipulatives do not inherently provide for non-rote learning. Many will immediately argue that this is a restatement of the obvious. This is not really the case, since the many questions of how manipulative aids are used by teachers and children to facilitate learning are complex, and the lack of precision in answers suggested by writers to date is greater than imagined.

Although an adult who considers the questions of how the manipulation of objects and symbolism used to record these manipulations are related may observe clearly the relationship, the same is far less clear for young children. The bridge between manipulation of objects to demonstrate mathematical concepts and the corresponding use of symbolism to record that manipulation is difficult to make. Moreover, there exists in the literature, to the writer's knowledge, no definitive means for accomplishing for children a meaningful relationship between the two modes of mathematical representation. While it is true that many capable children apparently are able to make this connection, the ability of average and below average children to do so is impressively weak.

For the purpose of communicating about mathematical ideas at least four modes of communication can be identified: oral, pictorial, manipulative, and symbolic. Questions about how one or more of the modes of learning teaching and communicating can be used to facilitate others have not been addressed. Some writers conjecture that there are readiness factors related to the ability to meaningfully employ symbolization. If so, are the readiness factors a matter of maturation or of education? Are some children's inabilities to deal with mathematical symbolism due to a deprived environment? Would one observe, through systematic observation or experimentation, that children whose early childhood was deprived of "normal" exposure to the symbolism communication represented by traffic

signs or cartoon strips, are the ones who experience the greatest difficulty with meaningful use of mathematical symbolization?

Linguists have identified distinct stages in children's understanding of sentence structure, both from the syntax and semantic structure of sentences. Eve Clark has reported interesting results about children's understanding of words by quantity--more and less in particular. Our observations within the second grade class in which we worked was that as late as second grade many children of average and below average intelligence make unusual responses to directions for changing a particular manipulative representative of a 2- or 3-digit number to be 1, 10, or 100 more or less. Many children exhibit difficulty about the relationship between "more" and "add to" and between "less" and "take away." That is, when asked to change a display to show "10 more," some children seem confused about whether to increase or decrease the display and also about whether to use a representative of 10 or 1. The questions of what and how extensive pre-number experience is necessary to develop facility with concepts of more, less, and one and ten more and less is a area for fruitful research in early childhood mathematics education. One wonders whether extensive experience with manipulative aids in the context of making sets of more, less, one more, and one less, etc. would be of help.

Also linguists have identified very orderly stages in children's linguistic development. To this writer's knowledge the question of whether teachers of mathematics regularly use linguistic structures which are comprehended by the age group has not been carefully investigated. For example, which of the question forms: "2 plus 4 is . . .", "2 plus 4 is what number?" Or "2 and 4 more is . . .," is best understood by first and second grade children? Such language usage should be investigated in the context of abstract mathematics and also in the context of manipulative activity. Teachers are observed to say, "Take five of the blocks away from the 7 units." Does the child hearing this pick out the words "5 . . . away . . . 7" and become confused about whether to take away 5 or take away 7? Would a more appropriate sentence structure be, "from your 7 blocks take away 5 blocks"?

Again thinking of the stages of linguistic development in children it seems that a comparable development of symbolic language might be observable. It might be the case that many children are expected to engage in uses of symbolic language representative of a "linguistic stage of development" well beyond where they are developmentally. Identification and characterization of such stages could prove beneficial to mathematics education.

Some children are observed to continue to have counting difficulties on into and beyond second grade. For example, when asked to count to determine the number represented by a Dienes blocks display of 3 longs and 4 units, some children are observed to count 10, 20, 30, 40, 50, 60, 70, or 1, 2, 3, . . ., 7 while touching each object in turn. It is possible via observation and experimentation, to determine whether such behavior is a result of inadequate counting skills, or perceptual problems or other learning handicaps? What are appropriate learning activities for children who exhibit these difficulties?

Some children have considerable difficulty acquiring mathematical concepts from pictures. For example, when shown a picture display of a Dienes blocks representation of the two addends 34 and 23, some children were observed to do the following in counting to determine the number represented altogether by the blocks:

> 10, 20, 30 (stopping and exhibiting behavior to suggest that he knew he should move to the next set of longs and count these, but continued counting instead)
>
> 10, 20, 30, 31, 32, . . . (stopping again and exhibiting similar behavior.)

What perceptual, maturational or other psychological phenomena explain such difficulty? Is it perhaps due to inability to perform multiple classifications? That is: Does the child once having the 3 longs for 34 classified as part of this set experience difficulty in reclassifying these longs into a class containing these 3 longs and the 2 longs for 23? Presumably, if a child had made this reclassification, he would be able to count 10, 20, 30, 40, 50, 51, . . ., 57.

An attempt was made in this teaching experiment to incorporate into the enactive phase of instruction in the 2M group the learning activity referred to by Wittrock as generative processing. As indicated herein, the proposed procedure apparently failed to accomplish this. It is, however, still a viable conjecture that such generative processing which requires the learner to go beyond imitative behavior with the manipulative aids may, in fact, be the significant variable associated with effective use of manipulative aids. This may be an essential activity with manipulative aids to force the child beyond imitative behavior, avoid rote activity with the manipulative, and enhance the abstraction which takes place from the manipulative embodiment. To test this conjecture it is suggested that investigation be conducted so that the enactive phase of instruction require children to: (1) Extend their ability to represent certain mathematical concepts from the manipulative aid first used in demonstration and practice to a "new" manipulative, and (2) Observe and explain the way(s) in which the new manipulative embodies the concepts, and whether or not and how the embodiment features are alike or different. For example, if a child learns via demonstration and practice (imitative behavior) to represent 34 as 3 longs and 4 units (Dienes blocks); then he would be required via generative processing to figure out how to represent 34 with another manipulative aid. Obvious variables are suggested at this point, the child can be expected to use generative processing to apply to a manipulative in the same category or to a manipulation in another category. New manipulative aids (and thus subsequent generative processing) would be introduced into the learning environment on a regular basis. Such generative processing could be considered within a uni- or multi - embodiment environment and the relative effectiveness compared.

The teaching groups for this teaching experiment were chosen so that within each group there existed an extreme range of ability, from very low to very high. The design of the experiment does not give indication about whether there are ability by type-of aid interactions, or ability by multiplicity-of-aid interactions, etc.

The content of this teaching experiment dealt with the concept of place value numeration and the use of this concept in addition and subtraction. Basic to the concept of place value numeration is the systematic grouping of objects. In addition, the ability to use a group (ten objects for base ten numeration) as an entity of one is essential. Related to this is the ability to observe that 10 object representatives of a unit equals 1 object representative of ten (i.e., that 1 ten = 10 ones). There are numerous investigable questions related to children's acquisition and understanding of this notion. For example, do children more readily accept 1 ten as equal to 10 ones when the representative for ten is decomposable into the 10 ones than when the representative for ten is not decomposable but is in linear measure equal to 10 ones or when it is neither decomposable nor equal in linear measure to 10 ones? What age and socioeconomic differences are observable concerning the previous questions? What experience differences are observable? Can children be taught so they internalize the concept of 1 ten as equal to 10 ones, or is it related to maturation? If it is related to maturation, what are some readiness experiences or indicators related to the acquisition of this motion?

APPENDIX A

SAMPLE OF INSTRUCTIONAL MATERIALS

U2 - Lesson 2 - Two-Digit Numerals - Enactive Level

Objectives: Learn block displays for two-digit numerals.
Children will give oral responses to teacher's manipulative display, and will give manipulative responses to numbers read orally by the teacher.

Materials: 40 longs and 40 units for each child.
40 longs and 40 units for the teacher.

Procedure: Follow the procedure suggested in CLASS ACTIVITIES.

CLASS ACTIVITIES:

By using questions, discussion, motions, direct command, or whatever is comfortable to you, DIRECT THE CHILDREN TO:

DO	
1. Display 9 units.	COUNT THE BLOCKS.
Display 10 units.	COUNT THE BLOCKS.
Do 1-1 matching of 9-set with subset of 10-set.	
Point to 10 units	TELL HOW MANY MORE HERE
Point to 9 units.	THAN HERE.
	TELL THAT TEN IS ONE MORE THAN NINE.
2. Display 10 units.	COUNT THE BLOCKS.
Display one long.	CALL THIS A LONG.
Line up the units along the long.	TELL HOW MANY IN A LONG.
Point to the long.	TELL THAT THE NUMBER IN A LONG
Point at the 10 units.	IS THE SAME AS TEN BLOCKS.
3. Display 12 units in one group.	COUNT THE BLOCKS AS YOU POINT.
Change the 12 units to one 10-set (counting as you do) and one 2-set.	
Point at the 10-set.	TELL HOW MANY BLOCKS ARE HERE.
Point at the 2-set.	AND HERE.
Replace the 10-set with a long.	TELL THAT TWELVE IS A SET OF TEN AND TWO MORE.

Point to the long..........................	TELL THAT TWELVE IS ONE LONG
Point to the 2-set..........................	AND TWO ONES.
	TELL THAT TWELVE IS ONE TEN AND TWO ONES.
4. Display 14 blocks as one 10-set and a 4-set..........................	COUNT ALL THE BLOCKS.
Display as one long and a 4-set..........................	TELL WHETHER THERE IS THE SAME NUMBER IN EACH DISPLAY.
Point at the long and 4-ones..........................	TELL HOW MANY ARE HERE.

5. Repeat 3 with sixteen, fourteen, eleven and twenty-three

6. Repeat 4 with sixteen, fifteen, and twenty-four

7. Give each child 40 longs and 40 units.	
Display one long and 4 ones..........................	SHOW THIS MANY WITH SINGLE BLOCKS.
Point to the long and the 4 ones..........................	TELL THAT ONE TEN AND 4 ONES IS FOURTEEN.
Display one long, and 7 ones..........................	SHOW THIS WITH SINGLE BLOCKS.
	TELL HOW MANY BLOCKS.
Point to the long..........................	TELL HOW MANY IN THIS LONG.
Point to the long..........................	TELL THAT THAT THERE IS ONE TEN
Point to the 7 ones..........................	AND SEVEN ONES.
Put out 2 longs and 3 ones..........................	PUT DOWN 2 LONGS AND 3 ONES.
Point to one long..........................	TELL HOW MANY THIS IS.
Point to next long..........................	AND THIS.
Point to both longs..........................	TELL HOW MANY TEN AND TEN IS.
Point to the ones..........................	TELL HOW MANY MORE ONES.
	TELL THAT ALTOGETHER THERE ARE TWENTY-SEVEN.

Put out 2 longs and 7 ones..................	PUT OUT 2 LONGS AND 7 BLOCKS.
Point to one long..........................	TELL HOW MANY THIS IS.
Point to the next long.....................	AND THIS.
Point to both longs........................	TELL THAT TEN AND TEN IS TWENTY.
Point to the longs and 7 ones..............	TELL HOW MANY ALTOGETHER.

Repeat this sequence with 3 longs and 2 ones.
2 longs and 0 ones.
1 long and 0 ones.

8.	PUT OUT 14 BLOCKS.
	MAKE THEM INTO A SET OF TEN AND 4 MORE.
Demonstrate.	
Point to one child's 10-set................	TELL THAT 10 BLOCKS IS THE SAME AS ONE LONG.
	REPLACE TEN BLOCKS BY ONE LONG.
Demonstrate.	
	TELL THAT FOURTEEN IS ONE TEN AND FOUR ONES.
	TELL HOW MANY ALTOGETHER.
	PUT OUT 23 BLOCKS.
Demonstrate.	
	MAKE A SET OF 10 BLOCKS.
Demonstrate.	
	MAKE ANOTHER SET OF 10 BLOCKS.
Demonstrate.	
Point to one 10-set........................	TELL THAT THOSE 10 ONES ARE THE SAME AS ONE LONG.
Point to 2nd 10-set........................	TELL THAT THOSE ARE THE SAME AS ONE LONG.

	TELL THAT TWENTY-THREE IS TWO TENS AND THREE ONES. TELL HOW MANY ALTOGETHER.

Repeat this sequence for 17.

Repeat this sequence for 27.

APPENDIX B

EVALUATOR'S SCRIPT FOR

INTERVIEW 1

TV

Write 35 on chart.

Read this for me. (1)

Turn page.

Write 53 on chart.

Read this for me. (2)

Interview

Write 53 on paper. You said this (point) is (___).

Why is it (___)?

Pointing to the 5 - What does the 5 mean?

Pointing to the 3 - What does the 3 mean? (3)

TV

Write 24 on chart.

I think of 24 like this -
twenty and four.

Now, you think of twenty-four in
some other way.

How are you thinking of twenty-four?

Interview

Hold up card 20 + 4 I can think of twenty-four like this
(point to card), twenty plus four.

Now, you think of twenty-four in
some other way.

TV

Have problem written on chart.

Give student a worksheet.

____ tens and ____ ones	Watch me, I will fill in the top line.
____ and ____	(Write 4 tens and 6 ones)
____	

Interview

Point to third line.	Why did you write (___) here.
Point to 1st blank second line.	Why did you write (___).
Point to 2nd blank second line.	Why did you write (___).

TV

Write 20 on chart.	
	Now, I want you to write the number which is ten more.
Turn page.	
Write 32 on chart.	
	Write the number which is one more.
Turn page	
Write 30 on chart	
	Write the number which is one ten more.

Interview

Hold up card 20	
Point to first student response.	Why is this ten more than this (point to card)?

Hold up card 32

Point to second response. — Why is this one more than this (point to card)?

Hold up card 30

Point to last response. — Why is this one ten more than this (point to card)?

TV

Have 3 boxes with 10 pieces of candy in each box on table.

Open each box and show to student. — See, there are 10 pieces of candy in this box.

Put 3 boxes in a bag. — I put 3 boxes of 10 candies in the bag.

Write on your paper, how many pieces of candy are in the bag.

Interview

Tell me, how many pieces of candy are in the bag?

How do you know?

TV

Display all aids

Point to the aids.

Choose one of these to show the number twenty-three.

Interview.

Let student select aid.

When student finishes, say

What number did she say?

Point to the student's work. Why does this show (____)?

TV

Display all aids.

Write 32 on board.

Point to aids.

I want you to use one of these to show this number.

Interview

Let student select aid.

When student finishes,

hold up card with 32. What number is this?

Point to student's work. Why does this show (____)?

Interview

Place 5 cups with 10 pieces of candy in each and 10 single pieces of candy on the table.

(a) Pointing with sweeping motion to aid — Show me with these 27.

successful, go to question (b)

unsuccessful, show 27.

(b) Pointing with sweeping motion to aid — Show me the number which is one more.

successful, go to (c)

unsuccessful, show 28.

(c) What is the number?

(d) Pointing to aid — Show me the number which is 10 more.

successful, go to (e)

unsuccessful, show 38.

(e) What is the number?

Numeration (2-digit) Video____

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Student____ School____ Interviewer____ Group____

(1) Read 35 as ____

Read 53 as ____

Why 53 (___) ____

5 means ____

3 means ____

(2) Thinks of 24 as ____

After 20 + 4, thinks ____

(3) ____ tens and ____ ones

____ (b) and ____ (c) Why (a) ____

____ (a) Why (b) ____

Why (c) ____

(4) ten more than 20: wrote____ Why____

one more than 32: wrote____ Why____

one ten more than 30: wrote____ Why____

Evaluation: T-1

Numeration (2-digit) Student ____________________

(5) How many pieces of candy? wrote____ said____

How knows ____________________

(6) Aid chosen S U B G A C

Configuration

____

Why 23 ____________________

(7) Aid chosen S U B G A C

Configuration

____

Number read ____________________

Why (______) ____________________

(8) Showed 27 - Yes No ______

Showed 28 - Yes No ______

Read 28 - Yes No ______

Showed 38 - Yes No ______

Read 38 - Yes No ______

APPENDIX C

EVALUATOR'S SCRIPT FOR

INTERVIEW 2

T-1 Evaluation 11: Addition, Subtraction, Order with 2-digit numbers.

Problems 1-6: 1) $\begin{array}{r} 5 \\ \underline{+23} \end{array}$ 2) $\begin{array}{r} 30 \\ \underline{+40} \end{array}$ 3) $\begin{array}{r} 40 \\ \underline{+23} \end{array}$ 4) $\begin{array}{r} 12 \\ \underline{+7} \end{array}$ 5) $\begin{array}{r} 46 \\ \underline{+23} \end{array}$ 6) $\begin{array}{r} 27 \\ \underline{+35} \end{array}$

Give the student the addition packet. Follow directions 1-4 below, in administering the first problem. When the student has completed problem one, turn the page and proceed with the other problems. Note: Direction 5 is used only with problems 5 and 6.

1. Read this problem.

2. Do the problem.

3. What is the answer?

4. Tell me how you got the answer.

5. Give the student the [illegible] used in his/her

giving [illegible] the __________ (one of these)

how [illegible] the problem.

Problems 7-[illegible]: 7) $\begin{array}{r} 37 \\ \underline{-4} \end{array}$ [illegible] $\underline{-6}$ [illegible] $\underline{-53}$ 10) [illegible] $\underline{-24}$

Give the [illegible] above for [illegible] used only [illegible] problems [illegible] and [illegible]

Problems 11-15.

Directions for these problems are given orally. No written symbolism is used.

11.a. Do this problem in your head: 24 plus 32.

b. Tell me how you got the answer.

12.a. Give the student a bag with 23 pencils. You have 23 pencils in the bag.

b. Show the bag with 5 pencils. I have 5 pencils in my bag.

c. How many pencils do we have together?

d. Tell me how you got the answer.

13.a. Give the student the bank with 32 cents, top closed. You have 32 cents in your bank.

b. Show the bank with [illegible] cents, top closed. I have [illegible] cents in my bank.

c. How many cents do we have together?

d. Tell me how you got the answer.

14.a. Do this problem in your head: 45 take away 32.

b. Tell me how you got the answer.

15.a. Show the bag with 2[illegible] blocks, top closed. This bag has 2[illegible] blocks.

b. Remove 4 blocks and show to student. How many blocks are left in the bag?

c. Tell me how you got the answer.

Problems 16-17.

Give the student a worksheet showing a picture of an aid used with his/her group. For groups M and 2M use an abacus and Dienes blocks with both problems. Follow the directions below.

1. Write a problem that goes with this pictu.e.

2. How can you tell that the problem goes with the picture?

Problems 18-19.

Give the student 6 cups with 10 beans in each cup, 10 loose beans and the problem packet.

1. Read the problem.

2. Show me with these how you work the problem.

3. What is the answ

Problems 20-21.

Give the student a worksheet showing the manipulative aid used with his/her group. For the M and 2M groups use the abacus for problem 20; sticks for problem 21.

1. Read the problem.

2. Does the problem go with the picture?

3.a. If yes; tell me how it goes with the picture.

b. If no; change something to make it right.

Problems 22-23,

Give the worksheet packet to the student. Point to bottom problem.

1. This problem is harder for me to work.

2. Can you tell me why I think it is harder?

Problems 24-25: 24) | 56 52 | 25) | 27 41 |

Give the student the order packet. Follow directions given below.

1. Read the numbers.

2. a. (24) Box the number which is more.

   b. (25) Circle the number which is less.

3. Tell me, how did you know that ________ was more (less)?

4. Write one of these (show card with > and <) here (point) to make it true.

5. (25 only) Give the students the manipulative aid(s) used in his/her group. Show me with the ________ (one of these) why ________ is less than ________.

APPENDIX D

EVALUATOR'S SCRIPT FOR

INTERVIEW 3

Individual Testing

Item 1. (1) Read this. (Tens | ones / 3 | 15)

(2) Use these (or one of these) to show this.

If no response: a. Show this (point to 3 tens)

b. Show this (point to 15 ones)

(3) Altogether these show what number?

(4) Can you make it so there are more tens?

(5) Explain it to me.

(6) What number is this now?

(Look for evidence of trade of ten ones for 1 ten)

Item 2. (1) Read this. Tens | ones / 4 | 2

(2) Use these (or one of these) to show this.

(3) ~~The (aids) show what number?~~

(4) Can you use these to show (number) in another way?

If no response: can you make more ones?

(5) The (aids) show what number?

Item 3. (1) Use these (or one of these) and show me how to work this problem. $\left(\begin{array}{r} 58 \\ +25 \\ \hline \end{array}\right)$

~~(2) What is the answer?~~

Item 4. Same as Item 3. $\left(\begin{array}{r} 54 \\ -37 \\ \hline \end{array}\right)$

Item 5. (1) Read this problem. $\left(\begin{array}{r} 47 \\ +29 \\ \hline \end{array}\right)$

(2) Do it.

(3) Read the answer.

(4) Tell me how you did it.

Item 6. Same as Item 5. $\left(\begin{array}{r} 75 \\ -38 \\ \hline \end{array}\right)$

Item 7. Materials: 6 non-transparent cups with covers, 10 beans in each cup, 20 loose beans, and one empty cup.

(1) There are ten beans in this cup, ten beans in this cup, etc.

(2) Here are some loose beans.

(3) Use the beans to work the problem.

(4) What is the answer? (+7)

Item 8. Same as Item 7. (32 − 7)

Item 9. (1) Read this problem (34 + 29 = 513)

(2) Is the answer right?

(3) Explain.

Item 10. Same as Item 9. (43 − 26)

If items 5 and 6 were done wrong, ask to use aids.

Item 5. Show each number with aids, then the sum

Item 6. (1) Show the first number
(2) Finish the problem with the aids

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APPENDIX E

EVALUATOR'S SCRIPT FOR

INTERVIEW 4

TEST No. 4 T_1 June 1976

Individual Testing

Item 1. (1) Read this.

(2) Use these (or one of these) to show this.

If no response: a. Show this (point to 2 hundreds)

b. Show this (point to 13 tens)

c. Show this (point to 4 ones)

(3) Altogether these show what number?

(4) Can you make it so there are more hundreds?

(5) Explain it to me.

(6) What number is this now?

(Look for evidence of trade of ten tens for one hundred)

Item 2. (1) Read this.

(2) Use these (or one of these) to show this.

(3) These (point to aids) show what number?

(4) Can you make it so there are more tens?

(5) These show what number?

Item 3. (1) Use these (or one of these) and show me how to work this problem.

(2) What is the answer?

Item 4. Same as Item 3.

Item 5. Same as Item 3.

Item 6. Same as Item 3.

Item 7. (1) Read this problem.

(2) Do it.

(3) Read the answer.

(4) Tell me how you did it.

Item 8. Same as Item 7.

<u>Item 9</u>. <u>Materials</u>: 16 large 100-bean bags

26 small 10-bean bags

32 single beans

(1) There are 100 beans in each of these large bags, 10 beans in each of these small bags and here are loose beans.

(2) Use the beans to work the problem.

<u>Item 10</u>. Same as Item 9.

<u>Item 11</u>. (1) Is this right?

(2) Explain this to me.

<u>Item 12</u>. (1) Do these (point to Card 12a) show this number (point to 156 on Card 12b)?

(2) Explain this to me.

<u>Item 13</u>. (1) What is ten more than 137?

(2) Explain.

<u>Item 14</u>. (1) What is ten less than 253?

(2) Explain.

<u>Item 15</u>. (1) This is 150 (point to 150 on Card 15).

(2) How many tens?

If no answer: Use the 10-bean bags to show me this.

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